On the existence of subgame perfect equilibria in dynamic games with discontinuous payoffs.

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Abstract

We prove that for a large class of perfect information dynamic games with discontinuous payoffs and infinite actions sets, a subgame perfect equilibrium exists. For example, a game belongs to our class if for every strategy profile \( s \) which is not a subgame perfect equilibrium, there is some player which has a profitable deviation \( d_i \) at some history \( h_{i-1} \), and if this improvement is stable with respect to small perturbations (1) of the path generated by \( s \) beginning at \( h_{i-1} \) and (2) of the deviated path generated by \( s \) and \( d_i \) at \( h_{i-1} \). Our existence result encompasses as a byproduct the existence results of Harris [10], Hellwig et al. [11] or Carmona [5], which treat the case of continuous payoffs. We explain why approximating the initial game by sequences of finite games, like in Hellwig et al.’s [11], is not efficient to prove our main result. We provide also several generalizations of our main result, together with examples and applications (for example to a class of quitting games).

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1 Introduction

Subgame perfect equilibrium (SPE) concept is the cornerstone of dynamic strategic models. A possible explanation for its importance in the literature is that it refines Nash equilibrium - the most frequently used stability concept in Game theory -, and that it is particularly well fitted to deal with dynamic models. As a consequence, it is often used as a prediction tool, for example in General Equilibrium, Game Theory, or in Network formation theory.

Its popularity also comes from the fact that its existence (in pure strategies) can be obtained through a simple backward induction argument, in every extensive-form game with:

1. Perfect and complete information,
2. A finite horizon,

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3. A finite number of actions at each period (e.g., [?]).

Many papers extends this existence result, relaxing one of the above property. For example, an important step is made by Fudenberg and Levine [9] who get the existence of a SPE for infinite horizon games, assuming some continuity property of payoffs. But, as shown in Harris [10], their argument can not be extended to the case of an infinite number of actions at each period. Yet, many naturel models in Economics require infinite action sets at each period: e.g., Cournot or Bertrand competition models ([6], [4]), Rubinstein competition model [20], Stochastic games [21], etc.

The first extensions to the case of infinite sets of actions at each period have been provided by Harris [10], then Reny, Hellwig and Robson [11], and more recently Carmona [5]. All these papers give different proofs of the existence of a SPE (in finite horizon games), but all of them use the assumption that the payoff functions are path-continuous, which means that they are continuous functions of the sequences of actions that are played: that is, given a sequence of actions that is chosen by the players, each player gets a payoff that is stable with respect to small modifications of the sequence of actions.

Surprisingly, there is no general existence proof of a SPE which allows some specific kind of discontinuities of the payoff functions. Yet, many games introduced in Economics have discontinuous payoff functions (e.g., timing games, price and spatial competitions, auctions, bargaining, preemption games or wars of attrition, etc.). In particular, a large literature connected to discontinuous games has been developed for more than 20 years (e.g., [7, 8], [5], [18], [3], [15], [19, 17], [2], [13], [14, 22], [22], [12], [16], etc.). But all the attention has been concentrated on normal form games, that is on the static case. In particular, one of the most used and studied result in this literature is Reny’s existence theorem [18], which proves the existence of a pure Nash equilibrium for normal form games with possibly discontinuous payoffs. The class of games considered by Reny, called better-reply secure games, encompasses many standard examples of economic literature, as auction or duopoly games.

A possible reason for the asymmetry in the existing results between the dynamic case and the static case may be technical: indeed, for normal form games, compactness and convexity of the strategy spaces are generally sufficient conditions to apply classical fixed-point theorems (like Brouwer or Kakutani’s one), in order to get the existence of a Nash equilibrium, at least for continuous payoff functions. In case of discontinuities, one can often approximate the payoff functions by continuous ones, which allows to apply the fixed-point approaches above to a sequence of approximated games, and then to use some limit argument to prove the existence of a Nash equilibrium.

For extensive-form games, there is, in general, no natural topology on the strategy spaces\(^1\) for which (1) the payoffs are continuous (with respect to profile of strategies) and (2) strategy spaces are compact, even when the payoff functions are path-continuous. As a consequence, given any well behaved extensive-form game \(\Gamma\), the normal form game \(G^{\Gamma}\) associated to this game does not possess, in general, sufficiently nice properties to make possible to apply a direct topological fixed-point approach to get the existence of a Nash equilibrium of \(G\), and SPE are in general even more difficult to obtained. On the other hand, it is always possible to approximate the action sets by finite ones, to consider a SPE of the finite approximation game by a standard backward induction scheme, and then to get a SPE of the initial game by a limit argument when the payoff functions are path-continuous (see [11]). But this approach may be inefficient

\(^1\)Recall that in a dynamic setting, we have to distinguish actions, that are chosen by players at each period, from strategies, which are chosen at the beginning of the game, as functions of all possible histories.
in case of discontinuities.

In this paper, we prove that for a large class of discontinuous perfect information games, a subgame perfect equilibrium exists. The main property which guarantees the existence of a SPE is called path-secure deviation property: roughly, it says that for every strategy profile \( s \) which is not a subgame perfect equilibrium, there is some player at some history \( h_{i-1} \) which has a profitable deviation \( d_i \), and such that this improvement is stable with respect to small perturbations (1) of the path generated by \( s \) beginning at \( h_{i-1} \) and (2) of the "deviated path" generated by \( s \) and \( d_i \) at \( h_{i-1} \). It has some similarity, in terms of definition, with the class of better-reply secure games introduced by Reny: recall a normal form game is better-reply secure if for every profile of strategies \( x \) which is not a Nash equilibrium and every profile \( u = (u_1, ..., u_n) \) of limit payoffs at \( x \) (i.e. a profile of payoffs which is a limit of payoff profiles \( u(x') \) when \( x' \) tends to \( x \)), some player \( i \) has a deviation \( d_i \) strictly above his limit payoff \( u_i \), and if this improvement is stable with respect to small perturbations of the other players’ strategies. But the connection between the two is informal, since none of the two models implies the other,\(^2\) and as a matter of fact, we will see that our method of proof is completely different from Reny’s one.

Then, in a second step, we obtain the existence of a SPE under properties that generalize path-secure deviation property, first by weakening the notion of perturbation considered, then by allowing sets of deviations. In particular, our existence results encompass as a byproduct the existence results of Harris [10], Hellwig et al. [11] or Carmona [5], which treat the continuous case.

The main idea of our proof is to associate to the game an object we call a net\(^3\) of finitely-deviated SPE paths: to build this object, we consider some finite approximation \( X' \) of the action sets, and some SPE profile \( s_{X'} \) restricted to \( X' \) (by a standard Backward induction argument). For every finite set \( D \) of "deviating" players and fixed strategies of these players (called deviations), we can then consider the path (i.e. sequences of actions of each player) for which each non-deviating player plays the strategy prescribed by \( s_{X'} \), and the deviating players play their deviation. Then, a net of finitely-deviated SPE paths is simply the nets generated by these paths, indexed by all families of finite approximations \( X' \) and all finite subset of deviations \( D \). The main advantage of this object is that (1) it represents in a suitable way the information conveys by the interdependent maximization behaviours of the players (2) it allows to approximate the extensive-form game not by a given finite approximation, as it is done in Hellwig et al. [11], but by any finite approximation, (3) and finally, SPE of the initial game should be related to limit points of this net (a limit point of a net being a generalization of a limit of a sequence).

Indeed, thanks to our main assumptions (path-secure deviation property or its generalizations), we will prove that for some well defined topology, any net of finitely-deviated SPE paths has a limit point which easily provide a SPE of the initial game.

The paper is organized as follows. In Section 2, we recall why, when the payoff functions are continuous and the action sets are infinite, the standard backward induction scheme cannot be used (following Hellwig et al. [11]), and explain the approximation method of these authors (Section 2.1). Then, we explain why their method fails in general when the payoffs are discontinuous, and illustrate this by an example (see Section 2.2). In Section 3, we define the model (Section 3.1), define path-secure deviation property (Section 3.2), together with the existence result of a SPE, with an example. A first extension (Section

\(^2\)Indeed, in our dynamic model, all player plays alternatively, although in the normal form games considered in Reny's paper, they play simultaneously.

\(^3\)Let us recall that a net is a "generalized" sequence that can be indexed by more general objects than integers.
3.3) is given, by restricting the notion of perturbation associated to the definition of path-secure deviation property. Then, a second extension (Section 3.4) is stated, which allows deviation sets in path-secure deviation property (instead of a single deviation). Again, these extensions are illustrated by examples. Last, in Section 4, we provide a class of games, we call quitting games, for which the existence of a SPE can be proved using the tools developed in the previous sections.

2 Existence of a SPE in continuous or discontinuous games:

some motivations

2.1 The continuous case

In [11], Hellwig, Leininger, Reny and Robson recall that the backward induction method does not extend, in general, to the case where players can choose an infinite number of actions. As an illustration, they consider a game with two players, denoted 1 and 3. Player 1 plays \( x_1 \in X_1 = [-1, 1] \), then player 3 plays \( x_3 \in X_3 = [-1, 2] \). The payoffs are \( u_1(x_1, x_3) = x_1 - x_3 \), \( u_3(x_1, x_3) = x_1 x_3 \) for every \((x_1, x_3) \in X_1 \times X_3\).

For every action \( x_1 \in X_1 = [-1, 1] \) of player 1, one can compute \( B_3(x_1) \) the set of best-responses of player 3 against \( x_1 \):

\[
B_3(x_1) = \begin{cases} 
-1 & \text{if } x_1 \in [-1, 0[ \\
[-1, 2] & \text{if } x_1 = 0 \\
2 & \text{if } x_1 \in ]0, 1[ 
\end{cases} \tag{1}
\]

Then, the backward-induction method prescribes to consider any selection \( b_3 : X_1 \rightarrow X_3 \) of \( B \) (which traduces that player 1 anticipates a rational behavior of player 3) and to solve the following maximization problem \((P)\) of player 1:

\[
(P) \max_{x_1 \in X_1} u_1(x_1, b_3(x_1)).
\]

Example 1. Graph of \( u_1(x_1, b_3(x_1)) \), \( x_1 \in [-1, 1] \), when \( b_3(0) = 0 \).

\[
\text{Figure 1: Payoff function } u_1(x_1, b_3(x_1)).
\]

\(^4\)which means that \( b_3(x_1) \in B_3(x_1) \) for every \( x_1 \in X_1 \).
Contrarily to what happens when strategy sets are finite, Problem ($P$) may have no solution, even when $u_1$ is assumed to be continuous, because $b_3$ can fail to be continuous (see Figure 1 above). It comes from the indeterminacy of the best-responses of player 3 when player 1 chooses the strategy $x_1 = 0$. In the example above, ($P$) has a solution if and only if $b_3(0) = -1$.

A first possible answer to this problem, mentioned in [11], is to consider the following "improved backward induction" principle: choose a particular selection $b_3$ which not only maximizes the payoff of player 3 (which is true by definition of $b_3$), but also maximizes the payoff of the player just before player 3 (here, player 1). This would reflect some "benevolent" behavior of player 3 to player 1. Formally, this is equivalent to consider a particular selection $b_3 : X_1 \to X_3$ of $B_3$ such that for every $x_1 \in X_1$, $b_3(x_1)$ is a solution of

$$\max_{b \in B_3(x_1)} u_1(x_1, b). \quad (2)$$

In the previous example, this implies that when $x_1 = 0$, Player 3 solves the indeterminacy in favor of player 1, by fixing $b(x_1) = -1$. If we impose this additional condition, then it is easy to see that despite the discontinuity of $b_3$ at $x_1 = 0$, the problem ($P$) has a solution: indeed, in this case, $u_1(x_1, b_3(x_1))$ is upper semicontinuous, thus has a maximum (here $x_1 = 0$) on the compact set $X_1$. In particular, this provides the unique subgame perfect equilibrium of the game: player 1 plays $x_1 = 0$, player 3 plays $-1$ if $x_1 \in [-1, 0]$ and plays $x_3 = 2$ if $x_1 > 0$.

Unfortunately, this "improved backward induction" approach fails with more than 2 players, as illustrated with the following modification of the previous game, proposed by the same authors in [11]. Player 2 is introduced in the new game, he moves between players 1 and 3 and his payoff is $u_2(x_2, x_3) = x_2 x_3$ with $x_2 \in X_2 := [-1, 2]$.

Then, the "improved Backward induction scheme" described above would induce player 3 to play a best reply $b_3(x_1, x_2)$ defined as follows:

$$b_3(x_1, x_2) = \begin{cases} -1 & \text{if} \quad x_1 \in [-1, 0] \\ [-1, 2] & \text{if} \quad x_1 = x_2 = 0 \\ 2 & \text{if} \quad x_1 = 0, x_2 > 0 \\ -1 & \text{if} \quad x_1 = 0, x_2 < 0 \\ 2 & \text{if} \quad x_1 \in [0, 1] \end{cases} \quad (3)$$

This would drive player 2 to choose the following best replies:

$$b_2(x_1) = \begin{cases} -1 & \text{if} \quad x_1 < 0 \\ 2 & \text{if} \quad x_1 \geq 0 \end{cases} \quad (4)$$

and player 1’s maximization problem would have no solution (see Figure 2 below). Yet, this game has a (unique) subgame perfect equilibrium, for which player 1 plays 0, player 3 plays 2 if $x_1 > 0$, $-1$ if $x_1 \leq 0$, and player 2 plays $-1$ if $x_1 \leq 0$ and 2 if $x_1 > 0$. At this subgame perfect equilibrium, player 3 is not benevolent to the player just before, but to player 1, and it is unclear how some general "benevolent" rule could drive to the existence of a subgame perfect equilibrium.

**Example 2.** Graph of $u_1(x_1, b_2(x_1), b_3(x_1, b_2(x_1)))$, $x_1 \in [-1, 1]$, when player 3 is benevolent to player 2.
To summarize, the "improved backward induction" rule above is inefficient to provide a subgame perfect-equilibrium in this game: because if player 3 wants to play in favor of player 2 when he is indifferent between several strategies, he could induce a behavior of player 2 which prevents player 1 to play at equilibrium.

In their paper, Reny et al. [11] propose the following answer: they approximate the action sets by finite ones, then consider some subgame perfect equilibrium of the finite approximation, obtained by standard backward induction. At the limit, when the finite approximating action sets converge to the initial game (for the Hausdorff distance), they are able to construct a SPE, \textit{up to small modifications of the limit strategies}. In other words, they use approximations of action sets by countable ones.

2.2 Discontinuous games

Is it possible to extend Hellwig et al. [11] approach for extensive-form games with discontinuous payoffs? The following example proves that in general, the answer is no:

\textbf{Example 3.} Consider a game with two players, denoted 1 and 2. Player 1 plays \( x \in X = [0, 1] \), then player 2 plays \( y \in Y = [0, 1] \). The payoffs are:

\[
\begin{align*}
    u_1(x, y) &= u_2(x, y) = 2y - 1 \quad \text{if} \quad x = 1 \\
    u_1(x, y) &= u_2(x, y) = -\frac{y^2}{2} \quad \text{if} \quad x \neq 1
\end{align*}
\]

This games does not have continuous payoffs with respect to the paths (because of the discontinuity for \( x = 1 \)). But there is a unique subgame perfect equilibrium of the game is: Player 1 plays \( x = 1 \), player 2 plays \( y = 0 \) if \( x \in [0, 1] \) and \( y = 1 \) if \( x = 1 \). In particular, this gives an equilibrium path \((1, 1)\), and a payoff vector \((1, 1)\). In addition, this SPE is highly credible: if Player 1 plays \( x = 1 \), then player 2 has to play \( y = 1 \), and even in case of small mistakes of player 2, the final payoff of player 1 will be is close to 1 (thus this is a robust payoff). Thus, player 1 has to play \( x = 1 \), since if he plays \( x < 1 \), he is certain to get less than 0.

The method of Reny, Robson and Hellwig (approximating the strategy spaces by some finite grid), which works for continuous games, may fail here. Indeed, consider the discretization \( X^n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \) and
\[ Y^n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}. \] This is a "good" approximation of \( X = Y = [0, 1] \) in the sense of Reny, Robson and Hellwig (because \( X^n \) and \( Y^n \) converges to \( X \) and \( Y \) for the Hausdorff distance), but the unique SPE path of the discretization is \((0, 0)\), which does not converge to the path \((1, 1)\) generated by the SPE of the initial game (contrarily to what happens in the proof of Hellwig et al [11]). Here, the issue is that the approximation can miss some important actions \((x = 1 \text{ and } y = 1)\), and considering strategies close to \((x = 1 \text{ and } y = 1)\) is not enough to keep the information of the payoffs at this point, because of the discontinuity. It is easy to modify our example (in the spirit of the previous examples) so that neither the standard backward induction principle nor the improved backward induction principle provide a SPE.

Thus, a natural question is to find some structural property of such a game that guarantees the existence of a SPE, despite the discontinuity: this is the aim of the next Section.

### 3 The general model, and the main existence result

#### 3.1 The model

We consider an extensive-form game with \( N \geq 2 \) players.\(^5\) Each player \( i \) is characterized by: (1) an action set \( X_i \), assumed to be a topological space\(^6\) and (2) a (possibly discontinuous) payoff function \( u_i : X_1 \times \ldots \times X_N \to \mathbb{R} \).

The rules are the following: player 1 chooses an action \( x_1 \) in \( X_1 \); then player 2, knowing \( x_1 \), chooses an action \( x_2 \), and so on for each actions \( x_i \) of player \( i = 3, \ldots, N \). At the end, i.e. when the last player \( N \) has chosen his action, each player \( i \) receives a payoff \( u_i(x_1, \ldots, x_N) \).

By definition, for every \( i = 0, \ldots, N-1 \), the set of histories for player \( i+1 \), denoted \( H_i \), is defined by

\[ \forall i \in N, \ H_i = X_1 \times \ldots \times X_i \]

and \( H_0 = \{x_0\} \), where \( x_0 \) is a given element (symbolizing the state before the game begins). In particular, an element of \( H_N \) is called a path. Throughout this paper, we will consider the following assumption:

**Compactness Assumption:** For every \( i \in N \), \( X_i \) is a compact topological set.

Remark that the payoff functions \( u_i : X_1 \times \ldots \times X_N \to \mathbb{R} \) are not assumed to be bounded. Throughout this paper, the set of histories \( H_i = X_1 \times \ldots \times X_i \) are endowed with the product topology (\( i = 1, \ldots, N \)). In particular, under Compactness Assumption, from Tychonov theorem, each \( H_i \) is compact (see Appendix 7 for some reminders about the product topology). Compactness assumption is standard in the relevant literature (e.g., [10] or [5]), but in general, the payoff functions \( u_i \) are also assumed to be continuous (thus bounded).

Now, it is convenient and standard to associate to the extensive-form game above the following normal form game, denoted \( G \), as follows:

- The set of players of \( G \) is \( N \).

\(^5\)For simplicity of notations, we will also denote \( N \) the set of players.
\(^6\)We do not require \( X_i \) to be Hausdorff.
A strategy $s_i$ of player $i \in N$ is a mapping from $H_{i-1} \to X_i$ which associates to every possible history $h_{i-1} \in H_{i-1}$ for player $i$ some action $s_i(h_{i-1}) \in X_i$. In particular, a strategy $s_1$ for player 1 can also be equivalently described as an element $x_1 \in X_1$ (an equivalent description that we will be sometimes used in this paper), even if it is formally a mapping from $\{x_0\}$ to $X_1$. For every $i \in N$, we let $S_i$ be the set of strategies of player $i$. Each set $S_i$ can be identified to $X_i^{H_{i-1}}$. It could be endowed with the product topology (then each $S_i$ would be compact from Tychonov theorem, since $X_i$ is compact (see Appendix 7)). But as a matter of fact, the topology on the $S_i$ will play no role in this paper.

To define the payoff functions of the normal-form game $G$ associated to the extensive-form game, fix a strategy profile $s = (s_1, ..., s_N) \in S$. This generates a path in $H_N$, denoted $p(s) = (x_1, ..., x_N)$, defined inductively by $x_1 = s_1, x_2 = s_2(x_1), ..., x_N = s_N(x_1, ..., x_{N-1})$. Then, the payoff function $U_i$ of player $i \in N$ in $G$ is defined by

$$U_i(s_1, ..., s_N) := u_i(p(s)).$$

The (standard) stability notion which will be considered in this paper is Subgame Perfect Equilibrium (SPE for short). Informally, this is a strategy profile $(s_1, ..., s_N)$ in $G$ such that for any given history $h_{i-1}$, no player $j$ after $h_{i-1}$ has some incentive to change the action prescribed by $s_j$, given that all players $i$ other than $j$ respects their strategy $s_i$ after $h_{i-1}$. To define SPE formally, for every $i \in N$ and for every history $h_{i-1} = (x_1, ..., x_{i-1}) \in H_{i-1}$, we now define a normal form game played after $h_{i-1}$, denoted $G(h_{i-1})$, as follows: first, if $i = 1$, that is $h_{i-1} = \{x_0\}$, then $G(h_{i-1}) = G$. Otherwise:

- The set of players of $G(h_{i-1})$ is $\{i, i+1, ..., N\}$.
- A strategy $s_j$ of player $j \in \{i, i+1, ..., N\}$ is a mapping from $\{(x_1, ..., x_{i-1})\} \times X_i \times ... \times X_{j-1}$ to $X_j$ for $j > i$, or from $\{(x_1, ..., x_{i-1})\}$ to $X_i$ for $j = i$.
- Every strategy profile $s = (s_1, ..., s_N)$ defines a unique path beginning at history $h_{i-1} = (x_1, ..., x_{i-1})$, denoted $p_{h_{i-1}}(s) = (x_i, ..., x_N)$, by the following inductive construction: $x_i = s_i(x_1, ..., x_{i-1}), x_{i+1} = s_{i+1}(x_1, ..., x_{i-1}, x_i), ..., x_N = s_N(x_1, ..., x_{N-1})$.
- Finally, we define a payoff function in $G(h_{i-1})$ for each player $j = i, ..., N$, denoted $U_j|_{h_{i-1}}$, by

$$U_j|_{h_{i-1}}(s_i, ..., s_N) = u_j(h_{i-1}, p_{h_{i-1}}(s))$$

for every strategy profile $s = (s_i, ..., s_N)$ of $G(h_{i-1})$.

- The game $G(h_{i-1})$ is called the subgame of $G$ beginning at $h_{i-1}$.

Note that for every $i \in N$, every $h_{i-1} \in H_{i-1}$, every strategy profile $s = (s_1, ..., s_N)$ induced a strategy profile also denoted (for simplicity) $(s_i, s_{i+1}, ..., s_N)$ on $G(h_{i-1})$.

Since SPE concept is built on the idea that no player has some profitable deviation along some path, it is convenient to introduce a notation to formalize such deviations (this notation will play an important role in the proof, and is not necessary in its general form to define SPE).
Definition 4. For every strategy profile \( s = (s_1, \ldots, s_N) \), \( i \in \{2, \ldots, N\} \) and given a history \( h_{i-1} \) and some fixed actions \( \bar{x}_{k_1}, \ldots, \bar{x}_{k_i} \) of players \( k_1 < k_2 < \ldots < k_l \in \{i, i+1, \ldots, N\} \), \( p_{|h_{i-1}}(\bar{x}_{k_1}, \ldots, \bar{x}_{k_i}, s_{-k_1-k_2-\ldots-k_l}) = (x_1, \ldots, x_N) \) denotes the path beginning after \( h_{i-1} \) and constructed inductively by following the prescription of \( s_i, s_{i+1}, \ldots, s_N \), except for stage \( k_1, \ldots, k_i \) where \( \bar{x}_{k_1}, \ldots, \bar{x}_{k_i} \) are played: that is, formally, \( x_i = s_i(h_{i-1}), x_{k_1} = s_{k-1}(h_{i-1}, x_i, \ldots, x_{k_1-2}), x_{k_1} = \bar{x}_{k_1}, x_{k_1+1} = s_i(x_1, \ldots, x_{k_1-1}, \bar{x}_{k_1}) \), \ldots, \( x_{k_2-1} = s_i(h_{i-1}, x_i, \ldots, x_{k_1-1}, \bar{x}_{k_1}, x_{k_1-1}, \ldots, x_{k_2-1}, x_{k_2-2}), x_{k_2} = \bar{x}_{k_2} \), and so on. When \( i = 1 \) and \( h_{i-1} = x_0 \), \( p_{|x_0}(\bar{x}_{k_1}, \ldots, \bar{x}_{k_i}, s_{-k_1-k_2-\ldots-k_l}) \) is simply denoted \( p(\bar{x}_{k_1}, \ldots, \bar{x}_{k_i}, s_{-k_1-k_2-\ldots-k_l}) \).

Throughout this paper, we shall denote by \( \Gamma = ((X_i)_{i \in N}, (u_i)_{i \in N}) \) the pair of action sets and payoff functions, defining the initial extensive-form game. The associated normal-form game described above will be denoted \( G = ((S_i)_{i \in N}, (U_i)_{i \in N}) \).

We can now formally define the basic equilibrium notion associated to any extensive-form game \( \Gamma \), called Subgame Perfect Equilibrium.

Definition 5. A strategy profile \( s = (s_1, \ldots, s_N) \) is a Subgame Perfect Equilibrium (SPE) of \( \Gamma = ((X_i)_{i \in N}, (u_i)_{i \in N}) \) if for every \( i \in N \) and every history \( h_{i-1} \), the induce strategy profile \( (s_i, s_{i+1}, \ldots, s_N) \) is a Nash equilibrium of \( G(h_{i-1}) \).

Since the game has a finite horizon, from the one-shot deviation principle, the condition above is well known to be equivalent to: for every \( i = 1, \ldots, N \), for every history \( h_{i-1} \in H_{i-1} \) and every deviation \( d_i \) of player \( i \),

\[
u_i(h_{i-1}, d_i, p_{|h_{i-1},d_i}(s)) \leq u_i(h_{i-1}, p_{|h_{i-1}}(s)),
\]

that is we can characterize SPE by only examining the one-shot games of every player at every possible history.

3.2 Path-secure deviation property

Let \( \Gamma = ((X_i)_{i \in N}, (u_i)_{i \in N}) \) be an extensive-form game. The following notion of "robust" profitable deviation plays a central role in the paper. For pedagogical reasons, we first present a simplified notion of robustness, together with an existence theorem of SPE. In the next subsections, we will present two different kind of extensions.

Definition 6. Player \( i \in N \) has a profitable path-secure deviation \( d_i \in X_i \) at some history \( h_{i-1} \), given a strategy profile \( \bar{s} \), if there exists some open neighborhoods \( V' \) of \((h_{i-1}, d_i, p_{|h_{i-1},d_i}(\bar{s})) \) and \( V \) of \((h_{i-1}, p_{|h_{i-1}}(\bar{s})) \) such that for every \( h'_{i-1} \in H_{i-1} \) and every \((y, y') \in \Pi^N_{j=i}X_j \times \Pi^N_{j=i+1}X_j \) such that \((h'_{i-1}, d_i, y') \in V' \) and \((h'_{i-1}, y) \in V \), we have

\[
u_i(h'_{i-1}, d_i, y') > u_i(h'_{i-1}, y) \tag{5}\]

At every history \( h_{i-1} \), any strategy profile \( \bar{s} \) and any deviation \( d_i \) of player \( i \) generate two paths, the path prescribed by \( \bar{s} \) beginning at \( h_{i-1} \), formally \((h_{i-1}, p_{|h_{i-1}}(\bar{s})) \), and the "deviated" path, formally \((h_{i-1}, d_i, p_{|h_{i-1},d_i}(\bar{s})) \), for which \( \bar{s} \) is followed from \( h_{i-1} \), except for player \( i \) who deviates and plays \( d_i \). Then, the assumption above says that \( d_i \) is a strictly profitable deviation for player \( i \) (from \( h_{i-1} \)) compared with the path prescribed by \( \bar{s} \), and this strict profitability is stable with respect to small
perturbations of the two previous paths. Importantly, remark that in Inequality (5), the perturbations $h'_{i-1}$ of the history $h_{i-1}$ in both sides are the same, and the deviation $d_i$ is not perturbed. Thus, the profitable deviation is secure under perturbations that modify slightly the two paths generated by the strategy profile $s$ and the deviation $d_i$, which explains the terminology.

From this notion, we derive the following structural property of extensive-form games:

**Definition 7.** An extensive-form game $G = (X, u)$ satisfies path-security deviation property if for every strategy profile $s$ which is not a SPE, some player $i$ has a profitable path-secure deviation $d_i$ in some subgame $h_{i-1}$.

**Remark 8.** If given the history $h_{i-1} \in H_{i-1}$ and given $\bar{s} \in S$, there exists $d_i \in X_i$ such that $u_i(h_{i-1}, d_i, p_{|h_{i-1}, d_i}(\bar{s})) > u_i(h_{i-1}, p_{|h_{i-1}}(\bar{s}))$ and such that $u_i$ is continuous at $(h_{i-1}, d_i, p_{|h_{i-1}, d_i}(\bar{s}))$ and at $(h_{i-1}, p_{|h_{i-1}}(\bar{s}))$, then clearly $d_i$ is a path-secure deviation at $h_{i-1}$ given $\bar{s}$.

Path-security deviation property guarantees the existence of SPE (see the proof in Appendix 5):

**Theorem 9.** If $G$ has the path-security deviation property, then it has a SPE.

**Example 10.** Consider a game with two players, denoted 1 and 2. Player 1 plays $x \in X = [0, 1]$, then player 2 plays $y \in Y = [0, 1]$. The payoffs are:

$$
\begin{align*}
    &u_1(x, y) = x, \quad u_2(x, y) = y \quad \text{if} \quad (x, y) \neq (0, 0), \\
    &u_1(0, 0) = u_2(0, 0) = -1.
\end{align*}
$$

The payoff functions of both players are not continuous at $(0, 0)$, yet the game satisfies path-security deviation property. Indeed, consider a strategy profile $s = (s_1, s_2)$ which is not a SPE. First case, assume $s_2(h_1)$ is not optimal for player 2 given some history $h_1 \in [0, 1]$. Thus $s_2(h_1) < 1$, and $d_2 = 1$ is a profitable path-secure deviation at $h_1$ given the strategy profile $s$: indeed, given the history $h_1$, $s$ generates a path $(h_1, s_2(h_1))$, and the deviated path when player 2 deviates for $d_2 = 1$ at the history $h_1$ is $(h_1, 1)$. Then, $u_2(h_1', x_2') \leq x_2' < u_2(h_1', d_2) = 1$ for every $(h_1', x_2')$ close enough to $(h_1, s_2(h_1))$, because $u_2(h_1', x_2')$ is equal to $-1$ or to $x_2'$, and because $s_2(h_1) < 1$. Second case, we now assume that player 2 plays optimally (i.e. plays $s_2(x_1) = 1$ for every $x_1 \in [0, 1]$) and that $s_1$ is not optimal, i.e. $s_1 < 1$. Then $d_1 = 1$ is a path-secure profitable deviation at $s$ for player 1: indeed, the path prescribed by $s$ is $(s_1, s_2(s_1)) = (s_1, 1)$ and the deviated path is $(d_1, s_2(d_1)) = (1, 1)$. Then $u_1(d_1, x_2'') = 1 > x'_1 \geq u_1(x'_1, x_2')$ for $(x'_1, x_2')$ close enough to $(s_1, s_2(s_1))$ and $x_2''$ close enough to $s_2(d_1) = 1$, because $u_1(x'_1, x_2'')$ is equal to $x'_1$ (which can be taken close enough to $s_1$ to ensure $x'_1 < 1$) or to $-1$.

A simple criterium to apply this theorem is the following property, which mixes upper semicontinuity and lower semicontinuity of the payoffs:

**(Usc-lsc secure deviation property):** for every strategy profile $s$ which is not a SPE, some player $i$ has strictly profitable deviation $d_i$ in some subgame $h_{i-1}$, such that $u_i(x)$ is u.s.c. with respect to $x = (x_1, ..., x_N)$ at $(h_{i-1}, p|_{h_{i-1}}(s))$, and is l.s.c. with respect to $x = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N)$ at $(h_{i-1}, d_i, p|_{h_{i-1}}(d_k, s_{-i})).$
Proof. If $s$ is not a subgame perfect equilibrium, from usc-lsc secure deviation property, there exists a SPE.

**Theorem 9.** Under usc-lsc secure deviation property, there exists a SPE.

**Example 12.** Consider a game with two players, denoted 1 and 2. Player 1 plays $x_1 \in [0, 1]$, then player 2 plays $x_2 \in [0, 1]$. The payoffs are:

$$
\begin{align*}
  u_1(x_1, x_2) &= 1 - x_1 & \text{if } (x_1, x_2) \neq (1, 0) \\
  u_1(1, 0) &= \frac{1}{2} \\
  u_2(x_1, x_2) &= -x_2 + x_1
\end{align*}
$$

The payoff function of player 1 is not continuous at $(1, 0)$. Yet the game satisfies the usc-lsc secure deviation property: consider some strategy profile $s$ which is not a SPE. First case, $s_2(x_1)$ is not optimal for player 2 for some $x_1 \in [0, 1]$, thus there exists $d_2$ such that $u_2(x_1, d_2) > u_2(x_1, s_2(x_1))$, and the usc-lsc secure deviation property is satisfied, because $u_2$ is continuous. Second case, we now assume that player 2 plays optimally (i.e. $s_2(x_1) = 0$ for every $x_1$) and that $x_1$ is not optimal, i.e. $x_1 > 0$. Then $d_1 = 0$ satisfies usc-lsc secure deviation property: indeed, the path of actions is $(x_1, 0)$, and the deviated path $(0, 0)$, and: (1) $u_1(d_1, 0) = 1 > u_1(x_1, 0)$, (2) $u_1(d_1, \cdot)$ is continuous (thus lower semicontinuous) with respect to the second variable, and (3) $u_1(\cdot, \cdot)$ is upper semicontinuous at $(x_1, 0)$ (and even continuous if $x_1 \neq 1$).

Another consequence of Theorem 9 (and also of Corollary 11) is the continuous case:

**Corollary 13.** ([10], [5], [11]) Every continuous game has the path-secure deviation property, and thus admits a SPE.

### 3.3 A first extension of Theorem 9

Let $\Gamma = (\langle X_i \rangle_{i \in N}, \langle u_i \rangle_{i \in N})$ be an extensive-form game. We first extend profitable path-security deviation definition, as follows. For simplicity, we keep the same terminology, and will precise in each context which definition of path-security we use.

**Definition 14.** Player $i \in N$ has a profitable path-secure deviation $d_i \in X_i$ at some history $h_{i-1}$ given a strategy profile $s$ if there exists some open neighborhoods $V'$ of $(h_{i-1}, d_i, p(h_{i-1}, d_i)(s))$ and $V$ of $(h_{i-1}, p(h_{i-1}(s)))$ and $X'$ a finite subgame of $X$, such that for every finite game $X''$ containing $X'$, every $h'_{i-1} \in X'_1 \times \ldots \times X''_{i-1}$ and every SPE $s = (s_1, \ldots, s_N)$ of $X''$ with $(h'_{i-1}, d_i, p(h'_{i-1}, d_i)(s)) \in V'$ and with $(h'_{i-1}, p(h'_{i-1}(s))) \in V$, we have

$$
  u_i(h'_{i-1}, d_i, p(h'_{i-1}, d_i)(s)) > u_i(h'_{i-1}, p(h'_{i-1}(s)) (6)
$$
The definition is very similar to Definition 6, except that the notion of neighborhood is now more restrictive, so that Inequality (6) is easier to get than the similar inequality in Definition 6. Again, let us recall that at every history $h_i$, any strategy profile $s$ and any deviation $d_i$ of player $i$ generate two paths, the path prescribed by $s$ beginning at $h_{i-1}$, formally $(h_{i-1}, p|_{h_{i-1}}(s))$, and the "deviated" path $(h_{i-1}, d_i, p|_{(h_{i-1}, d_i)}(s))$. Then, the property above says that player $i$ has some strictly profitable deviations $d_i$ at $h_{i-1}$ with respect to the path prescribed by $s$, which is immune to some kind of strong perturbations (more demanding than those used in Definition (6)): the perturbations involve small perturbations $h_i'$ of the history $h_i$ (this is unchanged with respect to the last definition), and small perturbations of the two paths above generated by any finite SPE $s$ with support "large enough" (this is the difference with the initial definition).

**Remark 15.** As explained above, if $d_i$ is a profitable path-secure deviation for player $i$ (in the sense of Definition 6) at some history $h_{i-1}$ given a strategy profile $s$, then it is true in the sense of Definition 14, although the converse implication can be false.

Path-secure deviation property is then defined exactly as in Definition 7, and similarly to the previous section, we get (see the proof in Appendix 5):

**Theorem 16.** Every game $G$ with the path-secure deviation property admits a SPE.

**Remark 17.** From the definition above, if $s$ is such that for $X''$ finite subgame of $X$ large enough, there is no $h_i'_{i-1} \in X'' \times \ldots \times X''_{i-1}$ and no SPE $s = (s_1, \ldots, s_N)$ of $X''$ with $(h_i'_{i-1}, p|_{h_i'_{i-1}}(s)) \in V$, then any $d_i \in X_i$ is a profitable path-secure deviation for player $i$ at $h_{i-1}$ given $s$: thus, to check path-secure deviation property, we can restrict to non SPE $s$ for which such $h_i'_{i-1}$ and $s$ exist.

**Example 18.** Let us prove that the game defined in Example 3 has the path-secure deviation property in the sense of Definition 14. Consider a strategy profile $(s_1, s_2)$ which is not a SPE. First assume that player 2 does not play optimally at some $x_1 < 1$ (thus $s_2(x_1) > 0$), then, $d_2 = 0$ is a strictly profitable path secure deviation in the subgame defined by $x_1$ (see Remark 8). Second, assume either that player 2 does not play optimally at $x_1 = 1$ (thus $s_2(1) < 1$), or that player 2 plays optimally everywhere and that player 1 does not play optimally (i.e $s_1 < 1$). In both cases, we can consider a finite game $X'$ which contains $x_1 = 1$ and $x_2 = 1$. For every SPE $(s'_1, s'_2)$ of every finite game containing $X'$, $s'_1 = 1$ and $s'_2(s'_1) = 1$. In particular, taking a neighborhood of $(s_1, s_2(s_1))$ small enough, $(s'_1, s'_2(s'_1))$ cannot be in $V$, and any deviation is path-secure profitable in the sense of Definition 14 (see Remark 17).

### 3.4 A second extension of Theorem 9

Let $\Gamma = ((X_i)_{i \in N}, (u_i)_{i \in N})$ be an extensive-form game. We now provide a second extension of path-secure deviation, for which we allow multivalued deviations:

**Definition 19.** Player $i \in N$ has a profitable path-secure deviation set $D_i \subset X_i$ at some history $h_{i-1}$ given a strategy profile $s$ if there exists some open neighborhoods $V' (d_i)$ of $(h_{i-1}, d_i, p|_{(h_{i-1}, d_i)}(s))$ for every $d_i \in D_i$ and $V$ of $(h_{i-1}, p|_{h_{i-1}}(s))$ such that: for every $h_i'_{i-1} \in H_{i-1}$ and every $(y, y') \in \Pi^N_{j=i+1} X_j$ such that $(h_i'_{i-1}, d_i, y') \in V'(d_i)$ for every $d_i \in D_i$ and $(h_i'_{i-1}, y) \in V$, we have

$$u_i(h_i'_{i-1}, d_i, y') > u_i(h_i'_{i-1}, y)$$

(7)
Here, in addition to Definition 6, the profitable deviation can depend (in a finite set) on the perturbation $h_i^{t-1}$ of the history and on the perturbations $y'$ and $y$ of the paths $p_{h_i^{t-1},d_1}(s)$ and $p_{h_i^{t-1},d_2}(s)$.

**Definition 20.** An extensive-form game $G = (X, u)$ satisfies path-security deviation set property if for every strategy profile $s$ which is not a SPE, some player $i$ has a profitable path-secure deviation sets $D_i$ in some subgame $h_i^{t-1}$.

Path-security deviation set property guarantees the existence of SPE (see the proof in Appendix 5):

**Theorem 21.** Every game $G$ with the path-security deviation set property admits a SPE.

The following example is a first illustration of Theorem 21.

**Example 22.** Consider a game with two players, denoted 1 and 2. Player 1 chooses $x_1 \in X_1 = [0,1]$, then player 2 chooses $x_2 \in X_2 = [0,1]$. The payoffs are:

$$
\begin{align*}
  u_1(x_1, x_2) &= 1, u_2(x_1, x_2) = 0 & \text{if } x_1 = \frac{1}{2} \text{ and } x_2 \notin \{0,1\} \\
  u_1(x_1, x_2) &= 1, u_2(x_1, x_2) = -1 & \text{if } [x_1 < \frac{1}{2} \text{ and } x_2 > 0 \text{ or if } [x_1 > \frac{1}{2} \text{ and } x_2 < 1] \\
  u_1(x_1, x_2) &= u_2(x_1, x_2) = 1 & \text{otherwise}.
\end{align*}
$$

Let us prove that this game has the path-security deviation set property. Consider a strategy profile $(s_1, s_2)$ which is not a SPE. First assume that player 2 does not play optimally at some $x_1 = s_1 \neq \frac{1}{2}$. Then clearly player 2 as a profitable path-secure deviation $d_2$ which is equal to $d_2 = 0$ if $x_1 < \frac{1}{2}$ and equal to $d_2 = 1$ if $x_1 > \frac{1}{2}$ (for example if $x_1 > \frac{1}{2}$, for any $h_1'\frac{1}{2}$ small perturbation of $x_1$, and any small perturbation $x_2'$ of $x_2 = s_2(s_1)$, we have $u_2(h_1', d_2 = 1) = 1 > u_2(h_1', x_2') = -1$.

Now assume that player 2 does not play optimally at $x_1 = s_1 = \frac{1}{2}$, thus $s_2(s_1) \notin \{0,1\}$. In this case, player 2 has a profitable path-secure deviation set $D_i = \{0,1\}$: indeed, for any $h_1' > \frac{1}{2}$ small perturbation of $x_1$, and any small perturbation $x_2'$ of $x_2 = s_2(s_1) = \frac{1}{2}$, we have $u_2(h_1', 0) = 1 > u_2(h_1', x_2') = -1$, and for any $h_1' > \frac{1}{2}$ small perturbation of $x_1$, and any small perturbation $x_2'$ of $x_2 = s_2(s_1) = \frac{1}{2}$, we have $u_2(h_1', 1) = 1 > u_2(h_1', x_2') = -1$. Last, if $x_1$ is not perturbed, $u_2(x_1, 1) = 1 > u_2(x_1, x_2') = 0$ for $x_2'$ close enough to $x_2$. This ends the proof, since player 1 always plays optimally.

Remark that this game does not possess path-security deviation property, because when $x_1 = s_1 = \frac{1}{2}$, there is no single profitable deviation robust in the sense of Definition 6.

4 Application to a class of quitting games and bargaining games

In this section, in addition to the previous examples, we define classes of games with discontinuous payoff functions for which our results can be applied.

**Example 23.** Consider the following class of games with $N$ players, we call quitting games: players $i = 1, 2, ..., N$ play alternatively and choose their actions $x_i \in X_i$, $i = 1, ..., N$, where each $X_i$ is assumed to be a compact topological set. Moreover, we assume properties (1) and (2) below:

1. Each player $i = 1, ..., N$ has a "quitting strategy" $q_i \in X_i$ which guarantees at least 0, that is $\inf_{x \in X} u_i(x_1, ..., x_{i-1}, q_i, x_{i+1}, ..., x_n) \geq 0$.

2. For every $x = (x_1, ..., x_N) \in X = \Pi_{i \in N} X_i$, $x_i \neq q_i$, such that $u_i$ is discontinuous at $x$, we have $u_i < 0$ on a neighborhood of $x$. 

13
(3) For every \( i \in N \) and \( x = (x_1, ..., x_N) \in X \), \( y \to u_i(y) \) is u.s.c. at \( y = (x_1, ..., x_{i-1}, q_i, x_{i+1}, ..., x_N) \) and \( (y_1, ..., y_{i-1}, y_{i+1}, ..., y_N) \to u_i(y_1, ..., y_{i-1}, q_i, y_{i+1}, ..., y_N) \) is continuous at \( (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N) \).

**Proposition 24.** Under assumptions (1), (2) and (3) above, the class of quitting game has the path-secure deviation property. In particular, it admits a SPE.

**Proof.** Consider a strategy profile \( s = (s_1, ..., s_N) \) which is not a SPE. Thus, there is some player \( i \) and some history \( h_{i-1} \) such that player \( i \) has a strictly profitable deviation \( d_i \) in the game beginning at \( h_{i-1} \), that is

\[ u_i(h_{i-1}, d_i, p(h_{i-1}, d_i)(s)) > u_i(h_{i-1}, p(h_{i-1})(s)). \]  

**First case:** assume \( u_i \) is not continuous at \( (h_{i-1}, p(h_{i-1})(s)) \) and \( s_i(h_{i-1}) \neq q_i \). Consequently, from Condition (2) above, we have \( u_i < 0 \) on some neighborhood of \( (h_{i-1}, p(h_{i-1})(s)) \). From Condition (1), there exists \( q_i \in X_i \) such that for every \( x = (x_1, ..., x_N) \in X \), \( u_i(x_1, ..., x_{i-1}, q_i, x_{i+1}, ..., x_n) \geq 0 \). In particular, the inequality

\[ u_i(h_{i-1}, q_i, p(h_{i-1}, q_i)(s)) > u_i(h_{i-1}, p(h_{i-1})(s)) \]  

is true, and remains true under small perturbations of \( (h_{i-1}, q_i, p(h_{i-1}, q_i)(s)) \) (\( q_i \) being fixed) and of \( (h_{i-1}, p(h_{i-1})(s)) \): that is, \( q_i \in X_i \) is a profitable path-secure deviation at \( h_{i-1} \), given \( s \).

**Second case:** assume \( u_i \) is not continuous at \( (h_{i-1}, p(h_{i-1})(s)) \) with \( s_i(h_{i-1}) = q_i \). Consequently, from Condition (1) above, \( u_i(h_{i-1}, p(h_{i-1})(s)) \geq 0 \). Thus, from Inequality (8) above, we get \( u_i(h_{i-1}, d_i, p(h_{i-1}, d_i)(s)) > 0 \), and since \( d_i \neq q_i \) this implies (from Condition (2)) that \( u_i \) continuous at \( (h_{i-1}, d_i, p(h_{i-1}, d_i)(s)) \). From Condition (3), \( u_i \) is u.s.c. at \( (h_{i-1}, p(h_{i-1})(s)) \), and in particular, Inequality (8) remains true under small perturbations of \( (h_{i-1}, d_i, p(h_{i-1}, q_i)(s)) \) and of \( (h_{i-1}, p(h_{i-1})(s)) \): that is, \( d_i \in X_i \) is a profitable path-secure deviation at \( h_{i-1} \), given \( s \).

**Third case:** assume \( u_i \) continuous at \( (h_{i-1}, p(h_{i-1})(s)) \). If \( u_i(h_{i-1}, p(h_{i-1})(s)) < 0 \), then as in the first case, we get that the quitting strategy \( q_i \) is a profitable path-secure deviation at \( h_{i-1} \), given \( s \). Thus, we can now assume \( u_i(h_{i-1}, p(h_{i-1})(s)) \geq 0 \). In particular, from Inequality (8) above, we get \( u_i(h_{i-1}, d_i, p(h_{i-1}, d_i)(s)) > 0 \), thus from Condition (2) we have to cases: \( u_i \) is continuous at \( (h_{i-1}, d_i, p(h_{i-1}, d_i)(s)) \), or \( d_i = q_i \), and in this case, from Condition (3), \( u_i \) is continuous at \( (h_{i-1}, q_i, p(h_{i-1}, d_i)(s)) \) with respect to \( (h_{i-1}, p(h_{i-1}, d_i)(s)) \) (i.e. \( q_i \) being fixed). In both cases, Inequality (8) above remains true under a small perturbations of \( (h_{i-1}, d_i, p(h_{i-1}, q_i)(s)) \) (\( d_i \) being fixed) and of \( (h_{i-1}, p(h_{i-1})(s)) \): thus, \( d_i \) is a profitable path-secure deviation at \( h_{i-1} \), given \( s \).

In all cases, we have proved that path-secure deviation property is satisfied.

The following example is an immediate by-product of the previous proposition:

**Example 25. (Oligopolistic competition)**

A single good is produced by \( N \) firms. The firms \( i = 1, 2, ..., N \) play alternatively and choose its production \( q_i \) (in some compact interval). Given the firms’ total output \( Q = q_1 + ..., q_N \), the market price \( p(Q) \) of the good is assumed to depend continuously on \( Q \).

The final payoff of each firm is \( u_i(q_1, ..., q_N) = q_i p(q_1 + ... + q_N) - C_q(q_i) - C_i^f \) if \( q_i > 0 \), with \( C_i(0) = 0 \), and \( u_i(q_1, ..., q_N) = 0 \) if \( q_i = 0 \). Here, \( C_i(\cdot) \) is the (non negative, continuous and variable) cost for firm \( i \), depending on the production \( q_i \), and \( C_i^f \geq 0 \) is a fixed cost, when firm \( i \) decides to produce the good.
The strategy \( q_i = 0 \) is a quitting strategy for each firm \( i \), and this game satisfies Conditions (1), (2) and (3) above: Condition (1) holds by definition of \( u_i \) for \( q_i = 0 \), Condition (2) is a consequence of \( u_i \) continuous at \( q \) when \( q_i \neq 0 \), and Condition (3) holds because \( u_i \) is continuous at \( q \) (in fact constant, equal to 0) with respect to \((q_1,\ldots,q_{i-1},q_{i+1},\ldots,q_N)\) when \( q_i = 0 \) is fixed, and u.s.c. at \( q \) with respect to \( q \) when \( q_i = 0 \).

In this game, the final payoff of firm \( i \) is discontinuous when \( q_i = 0 \), but Proposition 24 guarantees the existence of a SPE.

As a matter of fact, we can extend considerably our first class of examples:

**Example 26.** Consider the class of quitting games above, where each player \( i \) has a compact strategy set \( X_i \). We now assume the weaker properties (1) and (2) below:

1. For every player \( i \) and every history \( h_{i-1} \) there exists a finite set of strategies \( Q_i \subset X_i \) for player \( i \) (call local quitting strategies) and a neighborhood \( V \) of \( h_{i-1} \) such that for every \( h'_{i-1} \in V \) and every \((x_{i+1},\ldots,x_n)\in X_{i+1} \times \cdots \times X_n \), there exists \( q_i \in Q_i \) such that \( u_i(h'_{i-1}, q_i, x_{i+1},\ldots,x_n) \geq 0 \).
2. For every \( x = (x_1,\ldots,x_N) \in X = \Pi_{i\in N} X_i \) such that \( u_i \) is discontinuous at \( x \), at least (a) or (b) below are true:
   (a) \( u_i(x_1,\ldots,x_N) < 0 \) on a neighborhood of \((x_1,\ldots,x_N)\).
   (b) For every \( \varepsilon > 0 \), there is some neighborhood \( V \) of \((x_1,\ldots,x_{i-1})\) such that for every \((h'_{i-1}, h''_{i-1}) \in V \times V \), for every \((x'_{i+1},\ldots,x'_N) \in X_{i+1} \times \cdots \times X_N \), and every \((x''_1,\ldots,x''_N) \in X_1 \times \cdots \times X_N \), \( u_i(h'_{i-1}, h''_{i-1}, x'_{i+1},\ldots,x'_N) \geq u_i(h'_{i-1}, x''_1,\ldots,x''_N) - \varepsilon \).

Then under these assumptions, there exists a SPE.

Intuitively, (1) means that for every small perturbations of a given history \( h_{i-1} \), there is a finite set of ”quitting” strategies in \( Q_i \) that guarantees a non negative payoff (whatever the strategies chosen after \( i \)). (a) or (b) says that if \( x \) is a discontinuity point for player \( i \), then \( x \) should give a robust bad outcome for player \( i \) (Condition (a)) or gives almost the best one in a robust and strong sense.

**Remark 27.** In particular, Condition (b) implies that for every \((x'_{i+1},\ldots,x'_N) \in X_{i+1} \times \cdots \times X_N \) and every \((x''_1,\ldots,x''_N) \in X_1 \times \cdots \times X_N \), \( u_i(x_1,\ldots,x_i, x'_{i+1},\ldots,x'_N) \geq u_i(x_1,\ldots,x_i, x''_1,\ldots,x''_N) \), i.e. given history \((x_1,\ldots,x_{i-1})\), \( x_i \) is the best choice for player \( i \), for every choices after him.

**Proposition 28.** Under assumptions (1) and (2) above, the class of quitting games has the path-security deviation set property. In particular, it admits a SPE.

**Proof.** If \( s \) is not a SPE, then there is some player \( i \) and some history \( h_{i-1} \) such that player \( i \) has a strictly profitable deviation \( d_i \) in the game beginning at \( h_{i-1} \), that is

\[
 u_i(h_{i-1}, d_i, p_{\mid h_{i-1}, d_i}(s)) > u_i(h_{i-1}, p_{\mid h_{i-1}}(s)).
\]  

(10)

First case: assume \( u_i \) is not continuous at \((h_{i-1}, p_{\mid h_{i-1}}(s))\). From Inequality (10) above, Condition (b) cannot be satisfied at \((h_{i-1}, p_{\mid h_{i-1}}(s))\) (from Remark 27 above). Thus, from Condition (a), we have \( u_i(y) < 0 \) for every \( y \) in some neighborhood of \((h_{i-1}, p_{\mid h_{i-1}}(s))\). But from Condition (1), there exists a finite set of deviations \( Q_i \subset X_i \) and a neighborhood \( V \) of \( h_{i-1} \) such that for every \( h'_{i-1} \in V \) and for every \((x_{i+1},\ldots,x_n) \in X_{i+1} \times \cdots \times X_n \),
max_{q_i \in Q_i} u_i(h_{i-1}', q_i, x_{i+1}, ..., x_n) \geq 0.

Thus, in particular, for every $h_{i-1}' \in V$ and for every $(x_{i+1}, ..., x_n) \in X_{i+1} \times ... \times X_n$, for every $y$ in some neighborhood of $(h_{i-1}, p_{[h_{i-1}]}(s))$

$$\max_{q_i \in Q_i} u_i(h_{i-1}', q_i, x_{i+1}, ..., x_n) > u_i(y)$$

hence $Q_i$ is a profitable path-secure deviation set at $h_{i-1}$, given $s$.

Second case: now assume that $u_i$ is continuous at $(h_{i-1}, p_{[h_{i-1}]}(s))$. If it is also continuous at $(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s))$ then the strict inequality (10) remains true under small perturbation along the two paths $(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s))$ and $(h_{i-1}, p_{[h_{i-1}]}(s))$, thus $d_i \in X_i$ is a profitable path-secure deviation at $h_{i-1}$, given $s$. Thus, we can now assume that $u_i$ is not continuous at $(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s))$.

First subcase, assume (a) is satisfied at $(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s))$. In this case, we have the strict inequality $u_i(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s)) < 0$ and Inequality (10) implies also $u_i(h_{i-1}, p_{[h_{i-1}]}(s)) < 0$, and as above, Condition (1) guarantees the existence of a finite set of deviations $Q_i \subset X_i$ and a neighborhood $V$ of $h_{i-1}$ such that for every $h_{i-1}' \in V$ and for every $(x_{i+1}, ..., x_n) \in X_{i+1} \times ... \times X_n$, max$_{q_i \in Q_i} u_i(h_{i-1}', q_i, x_{i+1}, ..., x_n) > u_i(y)$ for every $y$ in some neighborhood of $(h_{i-1}, p_{[h_{i-1}]}(s))$ (because $u_i$ is continuous), and thus $Q_i \subset X_i$ is a profitable path-secure deviation set at $h_{i-1}$, given $s$.

We now examine the second subcase, for which case (b) is satisfied at $(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s))$. From inequality (10), and from Continuity of $u_i$ at $(h_{i-1}, p_{[h_{i-1}]}(s))$, there is $\varepsilon > 0$ such that for every $x'$ in some neighborhood of $(h_{i-1}, p_{[h_{i-1}]}(s))$,

$$u_i(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s)) > u_i(x') + \varepsilon. \quad (11)$$

From Condition (b) at $(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s))$, there is some neighborhood $V$ of $h_{i-1}$ such that for every $(h_{i-1}', h_{i-1}'', x_{i+1}', ..., x_N') \in V \times X$, for every $(x_{i+1}', ..., x_N') \in X_{i+1} \times ... \times X_N$ and every $(x_{i+1}'', ..., x_N'') \in X_{i+1} \times ... \times X_N$, $u_i(h_{i-1}', d_i, x_{i+1}', ..., x_N') \geq u_i(h_{i-1}'', x_{i+1}'', ..., x_N'') - \varepsilon$. In particular, taking $h_{i-1}'' = h_{i-1}$ and $(x_{i+1}'', ..., x_N'') = (d_i, p_{[(h_{i-1}, d_i)]}(s))$, we get $u_i(h_{i-1}', d_i, x_{i+1}', ..., x_N') \geq u_i(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(s)) - \varepsilon$. Taking into account Inequality (11) above, we finally get $u_i(h_{i-1}', d_i, x_{i+1}', ..., x_N') > u_i(x')$ for every $x'$ in some neighborhood of $(h_{i-1}, p_{[h_{i-1}]}(s))$, every $(x_{i+1}', ..., x_N') \in X_{i+1} \times ... \times X_n$ and every $h_{i-1}'$ in some neighborhood $V$ of $h_{i-1}$. Again, path-security deviation property is satisfied.

### 5 Appendix: Proof of Theorems 9, 16 and 21

We will prove Theorem 31 below, which generalizes Theorems 9, 16 and 21. It requires the two following definitions:

**Definition 29.** Player $i \in N$ has a weakly profitable path-secure deviation set $D_i \subset X_i$ at some history $h_{i-1} \in H_{i-1}$, given a strategy profile $\bar{s} \in S$, if there exists some open neighborhoods $V'(d_i)$ of $(h_{i-1}, d_i, p_{[(h_{i-1}, d_i)]}(\bar{s}))$ for every $d_i \in D_i$ and $V$ of $(h_{i-1}, p_{[h_{i-1}]}(\bar{s}))$, some finite subgame $X' \subset X$ such that for every finite game $X''$ containing $X'$, every $h_{i-1}' \in X_i' \times X''_{i-1}$ and every SPE $s = (s_1, ..., s_N)$
of $X''$ with $(h'_{i-1}, d_i, p_i(h'_{i-1}, d_i)(s)) \in V'(d_i)$ for every $d_i \in D_i$ and $(h'_{i-1}, p_i(h'_{i-1})(s)) \in V$, we have
\[
\max_{d_i \in D_i} u_i(h'_{i-1}, d_i, p_i(h'_{i-1}, d_i)(s)) > u_i(h'_{i-1}, p_i(h'_{i-1})(s))
\]

**Definition 30.** An extensive-form game $G = (X, u)$ has the weak path-security deviation set property if for every strategy profile $s$ which is not a SPE, some player $i$ has a weakly profitable path secure deviation set $D_i$ in some subgame $h_{i-1}$.

Similarly to Definitions 14 and 19, it says that if $s$ is not a SPE, some player $i$ has a strict improvement at some history $h_{i-1}$ which is robust to a specific kind of perturbations. The perturbations are very constrained (since they involve SPE of finite games "large enough" which generate paths or deviated paths that are close to the initial ones), and they incorporate both features of the perturbations of Definition 19 and Definition 14.

**Theorem 31.** For every extensive-form game $G = (X, u)$ which has the weak path-security deviation set property, there exists a SPE.

This theorem clearly generalizes Theorems 9, 16 and 21. We now give the proof.

**Step 1.** Let $D_0$ be the set of finite approximations of the game, that is, formally:
\[
D_0 = \{ \prod_{i=1}^{N} X'_i \subset X : \forall i = 1, ..., N, X'_i \text{ finite subset of } X_i \}.
\]

The inclusion $\subset$, restricted to $D_0$, is a reflexive and transitive binary relationship. In $(D_0, \subset)$, each pair $\Pi_{i=1}^{N} X'_i$ and $\Pi_{i=1}^{N} Y'_i$ has an upper bound $\Pi_{i=1}^{N} (X'_i \cup Y'_i)$ in $D_0$. Thus, the couple $(D_0, \subset)$ is a directed set (see Appendix 6 for reminders about directed sets).

**Step 2.** First, we define some useful notations: recall $X = X_1 \times ... \times X_N$ denotes the set of all possible paths, i.e. sequences of actions. For every fixed action $x_i \in X_i$ of each player $i \in \{1, ..., N\}$, we denote by $X_{|x_i}$ the set of all possible paths in $X$ for which player $i$ plays $x_i$, that is
\[
X_{|x_i} = X_1 \times ... \times X_{i-1} \times \{x_i\} \times X_{i+1} ... \times X_N.
\]

Similarly for every fixed strategies $x_{i_1} \in X_{i_1}$ of player $i_1$ and $x_{i_2} \in X_{i_2}$ of player $i_2$ (with $i_1 < i_2$), we denote by $X_{|(x_{i_1}, x_{i_2})}$ the set of all possible paths for which player $i_1$ plays $x_{i_1}$ and player $i_2$ plays $x_{i_2}$, that is
\[
X_{|(x_{i_1}, x_{i_2})} = X_1 \times ... \times X_{i_1-1} \times \{x_{i_1}\} \times X_{i_1+1} ... \times X_{i_2-1} \times \{x_{i_2}\} \times X_{i_2+1} ... \times X_N.
\]

Similarly, for every possible number of players $k = 0, ..., N$, every players $i_1 < i_2 < ... < i_k$ and every possible fixed strategies $x_{i_1} \in X_{i_1}$, $x_{i_2} \in X_{i_2}$, ... and $x_{i_k} \in X_{i_k}$ of these players, let us define $X_{|(x_{i_1}, x_{i_2}, ..., x_{i_k})}$ by
\[
X_{|(x_{i_1}, x_{i_2}, ..., x_{i_k})} = \{(y_1, ..., y_n) \in X : \forall j = 1, ..., k, y_{i_j} = x_{i_j}\}
\]

By convention, $X_{|(x_{i_1}, x_{i_2}, ..., x_{i_k})} = X$ for $k = 0$. 

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The sets $X_i(x_1, x_2, ..., x_k)$ are endowed with the product topology (and thus are compact): see Appendix 7.

Now, for every approximation of $X$, denoted $d = \prod_{i=1}^N X_i' \in D_0$, we can define an extensive-form game obtained by restricting the strategies of each player $i$ to $X_i'$, the payoff being obtained by considering the restriction of $u_i$ to $d$. Thus, given the profile of payoff functions $u$, we can identify $d$ with a (finite) extensive-form game. In particular, for every $d \in D_0$, there exists some strategy profile $s^d = (s_1^d, ..., s_N^d)$ which is a S.P.E. on the finite game $d$.

The paths generated by these strategy profiles defines a net $(p(s^d))_{d \in D_0}$ in the set of paths $X = X_1 \times ... \times X_N$ (see Appendix 6 for reminders about nets). Similarly, for every fixed strategy $x_i$ of player $i$, we can consider a net of paths where all players follows what is prescribed by $s^d$ except player $i$ who plays $x_i$, formally $(p(x_i, s_{-i}^d))_{d \in D_0}$ (following the notations introduced in Section 3.1). This defines a net with values in $X_i|_{x_i}$. We can do similarly when two players $i_1 < i_2$ have fixed strategies $x_{i_1} \in X_{i_1}$ and $x_{i_2} \in X_{i_2}$, and get a net of paths $(p(x_{i_1}, x_{i_2}, s_{-i_1,i_2}^d))_{d \in D_0}$, then do the same for 3 players, and so on until the case where the $N - 1$ first players have fixed strategies.

The following definition put together all the information conveyed by these paths, into one object:

**Definition 32.** A net of finitely-deviated SPE paths is, by definition,

$$(\pi^d)_{d \in D_0} = ((p(x_{i_1}, x_{i_2}, ..., x_{i_k}, s_{-i_1-i_2-...-i_k}^d))_{k = 0, ..., N-1, 1 \leq i_1 < i_2 ... < i_k \leq N-1, x_{i_1} \in X_{i_1}, ..., x_{i_k} \in X_{i_k})_{d \in D_0}$$

where by convention for $k = 0$, $p(x_{i_1}, x_{i_2}, ..., x_{i_k}, s_{-i_1-i_2-...-i_k}^d) = p(s^d)$.

Each term $\pi^d$ of this net belongs to the product set

$$\chi = X \times \prod_{i=1}^{N-1} X_i|_{x_i} \times \prod_{i=1}^{N-1} X_{i_1|_{x_{i_1}}} \times \prod_{i=1}^{N-1} X_{i_2|_{x_{i_2}}} \times ... \times \prod_{i=1}^{N-1} X_{i_{N-1}|_{x_{i_{N-1}}}},$$

where $\chi$, endowed with the product topology, is compact from Tychonov Theorem (See Appendix 7).

By compactness, the net of finitely-deviated SPE paths $(\pi^d)_{d \in D_0}$ above has a convergent subnet (see Appendix 6), that is there exists a set $D$ and a function $\phi : D \to D_0$ such that:

(a) For every $d \in D_0$, there exists $d \in D$ such that for every $d' \geq d$, $d \subset \phi(d')$.
(b) The subnet $(\pi_{\phi(d)})_{d \in D}$ is convergent (for the product topology defined on $\chi$).

In particular, given the definition of product topology, this implies that

(i) The net $(p(s_{\phi(d)}))_{d \in D}$ (which is the projection of $(\pi_{\phi(d)})_{d \in D}$ on the first component $X$ of $\chi$) converges to some path in $X$, denoted $p_{|\chi_0} = (p_{1|\chi_0}^1, p_{2|\chi_0}^2, ..., p_{N|\chi_0}^N)$.
(ii) For every $i = 1, ..., N-1$ and every $x_i \in X_i$, the net $(p(x_i, s_{-i}^d))_{d \in D}$ (which is the projection of $(\pi_{\phi(d)})_{d \in D}$ on the component $X_i|_{x_i}$ of $\chi$) converges to some path of $X_i|_{x_i}$, denoted $p_{|x_i} = (p_{1|\chi_0}^i, p_{2|\chi_0}^i, ..., p_{N|\chi_0}^i)$ (remark that $p_{1|\chi_0}^i = x_i$ by definition of $X_i|_{x_i}$). Roughly, $p_{|x_i}$ is a limit path for which the strategy player $i$ takes the fixed value $x_i$.
(iii) More generally, for every $k = 1, ..., N-1$, every $i_1 < i_2 < ... < i_k$ and every $x_{i_1} \in X_{i_1}, x_{i_2} \in X_{i_2}, ..., x_{i_k} \in X_{i_k}$, the net $(p(x_{i_1}, ..., x_{i_k}, s_{-i_1-i_2-...-i_k}^d))_{d \in D}$ (which is the projection of $(\pi_{\phi(d)})_{d \in D}$ on the
component \(X_i(x_{i1},...,x_{ik}) \text{ of } \chi \) converges to some path, denoted
\[
p_{i}(x_{i1},...,x_{ik}) = (p^1_{i|x_{i1},...,x_{ik}}, p^2_{i|x_{i1},...,x_{ik}}, ..., p^N_{i|x_{i1},...,x_{ik}}).
\]
This is the limit of the net of paths where all players \(i \) respect their strategy \(s_i^{\phi(d)} \) except the players \(i_1,...,i_k \) whose strategies \(x_{i1},...,x_{ik} \) are fixed.

**Step 3. Definition of the "limit" strategies \(\bar{s}_i \)**

The idea is to define, from the limit paths above \(p_{i}(x_{i1},...,x_{ik}) \), a tree that defines perfectly the strategies \(\bar{s}_i \) of all players (which will define the SPE). The difficulty is that two limit paths possess a lot of common histories: for example, when \(x_1 = p^1_{|x_0} \), the limit paths \(p_{|x_0} \) (for which no strategy is constrained) and \(p_{|x_1} \) (for which the first strategy \(x_1 \) is fixed) coincide at time 1, but in general they do not coincide after time 1. Thus, to avoid indeterminacy, we have to "remove" the path \(p_{|x_1} \) for \(x_1 = p^1_{|x_0} \), but also the path \(p_{|x_2} \) for \(x_2 = p^2_{|x_0} \), and so on. It can be done inductively as follows (at the same time, we define the limit strategies \(\bar{s}_i \)):

- Denote by \(N_0 \) the set of all possible non terminal histories along the path \(p_{|x_0} = (p^1_{|x_0}, p^2_{|x_0}, ..., p^N_{|x_0}) \), that is
\[
N_0 = \{p^1_{|x_0}, (p^1_{|x_0}, p^2_{|x_0}), ..., (p^1_{|x_0}, p^2_{|x_0}, ..., p^N_{|x_0})\}.
\]

We first define the strategies \(\bar{s}_1,...,\bar{s}_N \) on \(N_0 \cup \{x_0\} \) by simple continuation along this path: that is, \(\bar{s}_1(x_0) = p^1_{|x_0} \) (thus the strategy of player 1 is perfectly defined), and \(\bar{s}_j(p^1_{|x_0}, p^2_{|x_0}, ..., p^{j-1}_{|x_0}) = p^j_{|x_0} \) for every \(j = 2, ..., N \). Consequently, by construction, we get:
\[
p(\bar{s}) = p_{|x_0} = \lim_{d \in D} p(s^{\phi(d)}), \tag{13}
\]
i.e. the path \(p(\bar{s}) \) generated by the strategy profile \(\bar{s} \) is the limit path \(p_{|x_0} \) defined in Step 2 (i) above (remark that this is correct even if the strategies \(\bar{s}_i \) are not already defined everywhere)

- We now consider the set \(N_1 \) of all non terminal histories along limit paths \(p_{|x_i} \) (defined in Step 2, ii)) with one deviation \(x_i \) with respect to the path defined by \(N_0 \), that is formally
\[
N_1 = \bigcup_{i=1, ..., N-1, k=2, ..., N, x_i \in X_i} \{(p^1_{|x_i}, p^2_{|x_i}, ..., p^{k-1}_{|x_i})\} - N_0,
\]
where \(A - B \) denotes the the set of elements in \(A \) which are not in \(B \), also recalling that \(p^k_{|x_j} \) denotes component \(k \) of \(p_{|x_j} \). We can extend the strategies \(\bar{s}_j \) (for \(j = 2, ..., N \)) to \(N_1 \) by simple continuation along this path as follows: for every \(x_i \in X_i \) (\(i = 1, ..., N - 1 \)),
\[
\forall j = 2, ..., N - 1, \forall (p^1_{|x_i}, p^2_{|x_i}, ..., p^{j-1}_{|x_i}) \in N_1, \quad \bar{s}_j(p^1_{|x_i}, p^2_{|x_i}, ..., p^{j-1}_{|x_i}) = p^j_{|x_i}.
\]
In particular, this perfectly defines the strategy of player 2, since \(\bar{s}_2 \) is well defined on \(\cup_{p^1_{|x_0} \in N_1} p^1_{|x_0} = \cup_{x_i \neq p_{|x_0}} x_i \), and since \(\bar{s}_2(p^1_{|x_0}) \) has been defined in the previous point.

By construction, and from (ii) in Step 2, we have the formula: for every \(i = 1, ..., N - 1 \) and \(x_i \in X_i \),
\[
p(x_i, \bar{s}_{-i}) = p_{|x_i} = \lim_{d} p(x_i, s^{\phi(d)}_{-i}),
\]

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i.e. the path \( p(x_i, \bar{s}_{-i}) \) generated by \( \bar{s} \) (except for player \( i \) whose strategy is fixed equal to \( x_i \)) is the limit path \( p_{|x_i} \), defined in Step 2 (ii) above. Indeed, by construction, for every \( i = 1, \ldots, N-1 \), \( p(x_i, \bar{s}_{-i}) = (p_{1|x_i}, \ldots, p_{i-1|x_i}, x_i, p_{i+1|x_i}, \ldots, p_{N|x_i}) \). But we also have \( p(x_i, \bar{s}_{-i}) = (p_{1|x_i}, \ldots, p_{i-1|x_i}, x_i, p_{i+1|x_i}, \ldots, p_{N|x_i}) \), and from (13) above, \( p_{i|x_i} = \lim_{d} p\left(\phi_{i|d}(d)|x_i, \bar{s}_{-i} \right) = \lim_{d} p(x_i, \bar{s}_{-i}) = p_{|x_i} \) for every \( j = 1, \ldots, i-1 \) (because the path \( p(x_i, \bar{s}_{-i}) \) and the path \( p(s_{\phi(d)}) \) coincide before time \( i \)), and also \( x_i = \lim_{d} p(x_i, \bar{s}_{-i}) = p_{|x_i} \). Remark that it may be false that \( p_{x_0} = p(\bar{s}_{1}, \bar{s}_{-i}) \), that is, the limit path \( p_{x_0} = \lim_{d} p(s_{\phi(d)}) \) (where each player respect his strategy in \( s_{\phi(d)} \)) may be different from the limit of the paths where players \( 2,3, \ldots, N \) respect their strategies in \( s_{\phi(d)} \) and the strategy of player 1 is fixed, equal to his strategy \( \bar{s}_{1} = p_{x_0}^{1} \). This illustrates some of the difficulty of the proof.

Last, remark that by construction, \( \mathcal{N}_0 \cup \mathcal{N}_1 \) contains \( X_1 \).

- By induction, assume we have constructed some disjoint sets of histories \( \mathcal{N}_0, \mathcal{N}_1, \ldots, \mathcal{N}_k \) (for some \( k \in \{0, 1, \ldots, N-2\} \)) such that:
  1. Each \( \bar{s}_{i} \) is perfectly defined on (\( \{x_0\} \cup (\cup_{i=0}^k \mathcal{N}_i)) \cap (X_1 \times \ldots \times X_{i-1}) \) for every \( j = 1, \ldots, N-1 \),
  2. For every \( (x_{i_1}, \ldots, x_{i_k}) \in X_{i_1} \times \ldots \times X_{i_k} \),
     \[
     p(x_{i_1}, \ldots, x_{i_k}, \bar{s}_{-i_1-i_2-\ldots-i_k}) = \lim_{d} p(x_{i_1}, \ldots, x_{i_k}, s_{\phi(d)})
     \]
     (14)
  3. \( \cup_{i=0}^k \mathcal{N}_i \) contains all \( H_j \) for \( j = 1, \ldots, k \), i.e. all histories before \( k + 1 \). In particular, the strategies of players \( 1, \ldots, k+1 \) are perfectly defined.

We now consider the set \( \mathcal{N}_{k+1} \) of all histories along limit paths \( p_{|x_{i_1}, \ldots, x_{i_{k+1}}} \) with one deviation with respect to the paths defined by \( \mathcal{N}_k \), that is formally

\[
\mathcal{N}_{k+1} = \cup\{p_{1|h}, \ldots, p_{k|h}, p_{|h}, \ldots, p_{|h}^{N-1}\} \setminus (\cup_{i=0}^k \mathcal{N}_i)
\]

the union being taken for every \( i_1 < i_2 < \ldots < i_{k+1} \in \{1, 2, \ldots, N-1\} \), every \( x_{i_j} \in X_{i_j} \), \( x_{i_{k+1}} \in X_{i_{k+1}} \), with \( h = (x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}) \).

We then extend \( \bar{s}_j \) on \( \mathcal{N}_{k+1} \) for every \( j = k+2, \ldots, N \), by defining \( \bar{s}_j(p_{1|h}, p_{2|h}, \ldots, p_{k|h}, p_{|h}, \ldots, p_{|h}^{N-1}) = p_{j|h} \) for every \( (p_{1|h}, p_{2|h}, \ldots, p_{k|h}) \in \mathcal{N}_{k+1} \). As previously, this perfectly defined the strategy of player \( k+2 \), since \( \cup_{i=0}^{k+1} \mathcal{N}_i \) contains all possible histories in \( X_1 \times \ldots \times X_{k+1} \); indeed, if \( h = (x_1, \ldots, x_{k+1}) \) is a history in \( X_1 \times \ldots \times X_{k+1} \), either \( (x_1, \ldots, x_{k+1}) \in \cup_{i=0}^k \mathcal{N}_i \cup \cup_{i=1}^{k+1} \mathcal{N}_i \) (and \( \bar{s}_{k+2}(x_1, \ldots, x_{k+1}) \) is perfectly defined by induction), or \( (x_1, \ldots, x_{k+1}) \notin \cup_{i=0}^k \mathcal{N}_i \), and in that case, it means that all players \( i \) from 1 to \( k+1 \) deviates from the strategies prescribed by \( s_{\phi(d)} \) for playing \( x_i \). Thus, by definition, denoting \( h = (x_1, \ldots, x_{k+1}) \), we have \( (x_1, \ldots, x_{k+1}) \in (p_{1|h}, p_{2|h}, \ldots, p_{k|h}) \in \mathcal{N}_{k+1} \) (and \( \bar{s}_{k+2}(x_1, \ldots, x_{k+1}) = p_{k+2|h} \) is perfectly defined).

By construction, we get, for every \( (h, \bar{s}_{-i_1-i_2-\ldots-i_k-i_{k+1}}) \in \mathcal{N}_{k+1} \), where \( h = (x_{i_1}, \ldots, x_{i_{k+1}}) \),

\[
 p(h, \bar{s}_{-i_1-i_2-\ldots-i_k-i_{k+1}}) = p_{|h} = \lim_{d} p(h, s_{\phi(d)})
\]

Indeed, first remark that the first \( j \) components of the path \( p(h, \bar{s}_{-i_1-i_2-\ldots-i_k-i_{k+1}}) \) and of the path \( p(h, \bar{s}_{-i_1-i_2-\ldots-i_k}) \) coincide for every \( j < i_{k+1} \), since they are defined similarly as limits of path where all players between 1 and \( j \) respect their strategies in \( s_{\phi(d)} \) except players \( j_1, \ldots, i_{k+1} \) who plays \( x_{i_1}, \ldots, x_{i_{k+1}} \). In particular, for every \( j < i_{k+1} \), \( p(h, \bar{s}_{-i_1-i_2-\ldots-i_k-i_{k+1}}) = p(h, \bar{s}_{-i_1-i_2-\ldots-i_k} \bar{s}_{-i_1-i_2-\ldots-i_k}) \) which is also equal
to \( \lim_d p^j(x_{i_1}, ..., x_{i_k}, s_{\bar{x}_{i_1}-...-i_k}) \) by induction assumption. But this is also \( \lim_d p^j(h, s_{\bar{x}_{i_1}-...-i_k}) \) (always for the same reason that \( x_{ik+1} \) plays no role before time \( j < i_{k+1} \)), thus Equation (15) is true for \( j < i_{k+1} \). Now, for \( j = i_{k+1}, p^{k+1}(h, \bar{s}_{i_1}-...-i_{k+1}) = x_{i_{k+1}} \) since we have \( p^{k+1}(h, s_{\bar{x}_{i_1}-...-i_{k+1}}) = x_{i_{k+1}} \). Last, for \( j > i_{k+1} \), by definition, \( p^j(h, \bar{s}_{i_1}-...-i_{k+1}) = p^j_h \) which is also equal to \( \lim_d p^j(h, s_{\bar{x}_{i_1}-...-i_{k+1}}) \).

Finally, by induction, the strategies \( \bar{s}_i \) of all players are well defined (because they are defined on \( \{x_0\} \cup (\cup_{i=0}^{N-1} \mathcal{N}_i) \), the set of all non terminal possible histories), and we have, for every \( k = 0, ..., N-1, \) every \( 1 < i_1 < i_2 < ... < i_k < N-1 \) and for every \( (x_{i_1}, ..., x_{i_k}) \in X_{i_1} \times ... \times X_{i_k} \),

\[
p(x_{i_1}, ..., x_{i_k}, s_{\bar{x}_{i_1}-...-i_k}) = \lim_d p(x_{i_1}, ..., x_{i_k}, s_{\bar{x}_{i_1}-...-i_k}) \tag{16}
\]

where in particular \( p(\bar{s}) = \lim_d p(s_{\bar{x}}(d)) \) when \( k = 0 \).

**Step 4:** \( \bar{s} \) is a SPE by contradiction.

If the strategy profile \( \bar{s} = (\bar{s}_1, ..., \bar{s}_N) \) built in Step 3 is not a SPE, then applying weak path-security deviation set property, some player has a weakly profitable path secure deviation set \( D \) (which is also equal to \( \lim_s V \bar{\phi} \)). We will simply prove that this contradicts (for \( d \in D \) "large enough") that \( s_{\bar{x}}(d) \) is a SPE of the finite approximation \( \bar{\phi}(d) \) of \( X \). Here, weak path-security deviation set property means that there exist some open neighborhoods \( V^j(x) \) of \( (h_{i-1}, x^k, p_{(h_{i-1}, x^k)}(\bar{s})) \) for every \( x^k \in D_i \) and \( V \) of \( (h_{i-1}, p|h_{i-1}(\bar{s})) \), and there exist some finite approximation \( X^n \) of \( X \) such that for every SPE \( s = (s_1, ..., s_N) \) of any finite approximation of \( X \) containing \( X^n \), if \( (h_{i-1}', x^k, p_{(h_{i-1}', x^k)}(\bar{s})) \in V^j(x^k) \) for every \( x^k \) and \( (h_{i-1}', p|h_{i-1}(\bar{s})) \in V \), then

\[
\exists x^k \in D_i : u_i(h_{i-1}', x^k, p_{(h_{i-1}', x^k)}(\bar{s})) > u_i(h_{i-1}', p|h_{i-1}(\bar{s})) \tag{17}
\]

Now, \( h_{i-1} \in \{x_0\} \cup (\cup_{m=1}^{N-1} \mathcal{N}_m) \) (because by construction, this set contains \( X_1 \times ... \times X_{i-1} \)). Thus, \( h_{i-1} \) is in some \( \mathcal{N}_m \) for \( m = 1, ..., i-1 \) or is equal to \( x_0 \); thus, it can be generated by the strategy profile \( \bar{s} \) and by deviations (which does not respect the prescription of \( \bar{s} \)) in the sense that there exists \( y_{i_1} \in X_{i_1}, ..., y_{i_m} \in X_{i_m} \) (with \( 0 \leq i_1 < i_2 < ... < i_m < i \)) such that the history \( h_{i-1} \) can be written

\[
h_{i-1} = (p^1(y_1, ..., y_{i_m}, \bar{s}_{\bar{x}_{i_1}-...-i_m}), ... , p^{i-1}(y_1, ..., y_{i_m}, \bar{s}_{\bar{x}_{i_1}-...-i_m})) \tag{18}
\]

we recall that it means that \( h_{i-1} \) is the path from time 1 to time \( i-1 \), where each player \( i \) respect his strategy \( \bar{s}_i \), except for players \( l \) (\( l = 1, ..., m \)) who follow their fixed action \( y_{i_l} \). In particular, components \( i_1, i_2, ..., i_m \) of \( h_{i-1} \) are equal to \( y_{i_1}, y_{i_2}, ... \) and \( y_{i_m} \). Here, we allow \( m = 0 \), and in this case \( h_{i-1} = (p^1(\bar{s}), ..., p^{i-1}(\bar{s})) \).

Thus, we have

\[
(h_{i-1}, p|h_{i-1}(\bar{s})) = p(y_1, ..., y_{i_m}, \bar{s}_{\bar{x}_{i_1}-...-i_m}) = \lim_{d \in D} p(y_1, ..., y_{i_m}, s_{\bar{x}_{i_1}-...-i_m}) \tag{19}
\]

the last limit being a consequence of Equation (16).

For every deviation \( x_i \in X_i \) with respect to the action prescribed by \( \bar{s}_i \) at \( h_{i-1} \) (i.e. \( x_i \neq \bar{s}_i(h_{i-1}) \)), since \( (h_{i-1}, x_i) \) belong to \( \mathcal{N}_{m+1} \) (by definition of this set), we have also, again from Equation (16):

\[
(h_{i-1}, x_i, p|h_{i-1}, x_i(\bar{s})) = p(y_1, ..., y_{i_m}, x_i, \bar{s}_{\bar{x}_{i_1}-...-i_m}) = \lim_{d \in D} p(y_1, ..., y_{i_m}, x_i, s_{\bar{x}_{i_1}-...-i_m}) \tag{19}
\]
Let us consider the finite approximation $X'_1 \times \ldots \times X'_N$ of $X$ such that each $X'_k$ (for $k = 1, \ldots, N$) contains all the projections on $X_k$ of all possible paths defined by $D_i$, $h_{i-1}$ and $\bar{s}$ (that is, if $h_{i-1}$ is denoted $(h_{i-1}^1, \ldots, h_{i-1}^m)$, inductively, $X'_k = \{h_k^i\}$ for $k < i$, $X'_i = \{\bar{s}_i(h_{i-1})\} = D_i$, $X'_{i+1} = \{\bar{s}_{i+1}(h_{i-1}, x_i), x_i \in X'_i\}$ etc. In particular, $y_{i} \in X'_{i+1}$, $y_{i} \in X'_{i+1}$ and $D_i \subset X'_i$.

From (18) and (19) above, from the fact that $V'(x^k)$ is an open neighborhood of $(h_{i-1}, x_k, p(h_{i-1}, x^k)(\bar{s}))$ for every $x^k \in D_i$ and $V$ is an open neighborhood of $(h_{i-1}, p(h_{i-1}, x^k)(\bar{s}))$, and from Property (a) satisfied by the mapping $\phi$ (see Step 2), there exists $\bar{d} \in D$ "large enough" such that for every $d \geq \bar{d}$, the finite approximation $\phi(d)$ of $X$ contains $X'_1 \times \ldots \times X'_k$ and $X''$ (defined at the beginning of Step 4), and such that:

1. $p(y_{i1}, \ldots, y_{im}, s_{i1}, \ldots, t_{im}) \in V$.
2. For every $x^k \in D_i$, $p(y_{i1}, \ldots, y_{im}, x_k, s_{i1}, \ldots, t_{im}) \in V'(x^k)$.

Define $h'_{i-1} = (p_{i1}, \ldots, y_{im}, s_{i1}, \ldots, t_{im})$. Since each action $y_{i1}$ belong to $X'_{i1}$ for $l = 1, \ldots, m$ (by definition of $X'_{i1}$), since $\phi(d)$ contains $X'_1 \times \ldots \times X'_N$, and since $s_{i1}$ is defined on the finite game $\phi(d)$, we get that $h'_{i-1}$ belong to the finite game $\phi(\bar{d})$. In particular, rewriting (1) and (2) above, we get:

1. $p(h'_{i-1}, \bar{d})(s_{i1}(\bar{d})) \in V$.
2. For every $x^k \in D_i$, $p(h'_{i-1}, x_k, s_{i1}(\bar{d})) \in V'(x^k)$.

From (17) applied to $s = s_{i1}(\bar{d})$ and $h'_{i-1}$ (which is possible from (1') and (2') above, and because $\phi(d)$ contains $X''$ by definition), we obtain that there exists $x^k \in D_i \subset X'_i$ such that:

$$u_i(h'_{i-1}, x_k, p(h'_{i-1}, x_k, s_{i1}(\bar{d}))) = u_i(h'_{i-1}, x_k, p(h'_{i-1}, x_k, s_{i1}(\bar{d}))).$$

In particular, $x^k$ is a strictly profitable deviation for player $i$ (compared with $s_{i1}(\bar{d})(h'_{i-1})$) in the finite subgame $\phi(\bar{d})$ beginning at $h'_{i-1}$ (recall $x^k \in D_i \subset X'_i$ is an admissible strategy in $\phi(\bar{d})$).

This finally contradicts that $s_{i1}(\bar{d})$ is a SPE of $\phi(\bar{d})$, and ends the proof.

### 6 Appendix: Reminders about nets

By definition, a **direction** $\geq$ on a set $D$ is a reflexive, transitive and binary relation on $D$, with the property that each pair of $D$ has an upper bound. That is, for each pair $(\alpha, \beta)^2 \in D \times D$, there exists $\gamma \in D$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. Note that a direction is not required to be antisymmetric, thus it may not be a partial order. A **directed** set is, by definition, any set $D$ equipped with a direction $\geq$. A **net** in a set $X$ is a function $x : D \to X$, where $D$ is a directed set, called the **index set** of the net, and the elements of $D$ are **indexes**. This net is denoted $(x_\alpha)_{\alpha \in D}$, or simply $(x_\alpha)$. An element $x \in X$ is called a **limit** of the net $(x_\alpha)_{\alpha \in D}$ (with values in some topological space $(X, \tau)$) if for each neighborhood $V$ of $x$, there exists some index $\alpha_0$ (depending on $V$) such that for every $\alpha \geq \alpha_0$, $x_\alpha \in V$. It is denoted $x_\alpha \xrightarrow{\tau} x$.

A subnet of a net $(x_\alpha)_{\alpha \in D}$ is, by definition, a net $(y_\beta)_{\beta \in D'}$ such that there exists a function $\phi : D' \to D$ with:

1. For every $\beta \in D'$, $y_\beta = x_{\phi(\beta)}$.
2. For every $\alpha \in D$, there exists $\tilde{\beta} \in D'$ such that for every $\beta \geq \tilde{\beta}$, $\phi(\beta) \geq \alpha$.

An element $x \in X$ is called a **limit point** of the net $(x_\alpha)_{\alpha \in D}$ if $x$ is a limit of some subnet of $(x_\alpha)_{\alpha \in D}$.
Proposition 33. A topological space X is compact if and only if every net in X has a limit point.

7 Appendix: Reminders about product topology

Let \( X = \prod_{i \in I} X_i \) denote the Cartesian product of a family of topological spaces \( \left( (X_i, \tau_i) \right)_{i \in I} \). For \( j \in I \), the projection \( P_j : X \to X_j \) is defined by \( P_j(x) = x_j \) where \( x = (x_i)_{i \in I} \in X \). The product topology \( \tau \), denoted \( \prod_{i \in I} \tau_i \), sometimes called the Tychonoff topology, is the weakest topology on \( X \) that makes all projections \( P_j \) continuous.

Proposition 34. Consider a net \( (x^\alpha)_{\alpha \in D} \) in \( X \), where \( x^\alpha = (x^\alpha_i)_{i \in I} \). Then \( x^\alpha \xrightarrow{\tau} x := (x_i)_{i \in I} \) if and only if \( x^\alpha_i \xrightarrow{\tau_i} x_i \) for every \( i \in I \).

Tychonoff Theorem. The product of a family of topological spaces is compact for the product topology if and only if each factor is compact.

References


