Equilibria, fixed point, computation

Philippe Bich, PSE and University Paris 1 Pantheon-Sorbonne, France.

0. Presentation, syllabus

- Modelization in Micro, Macro, Network theory, game theory: what are we talking about ?
- A model in Economics, social science, ...: mathematical equations describing the evolution of an economical system (with agents).
- The agent can be a homo œconomicus (neo-classical paradigm: rational agent), or not (lack of rationality: cognitive limits, prudence, altruism, cognitive bias...)
- But his behaviour is influenced by some forces (happiness, social forces, altruism, ...) which can sometimes be contradictory.
- In a model, before studying the dynamic, interesting to study the stability of the system.
- Now, Equilibria (or fixed-point) comes on the scene!

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But in Physics, there are very stable principle (Nature minimizes energy...)

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Example 1: In game theory, Nash equilibrium.

Who is Nash ?

John Nash (American mathematician 1928-2015).

One of the greatest genius of mathematics according to some mathematicians.

Won Nobel prize in Economics and Abel medal (like Nobel prize in mathematics).

Was ill (schizophrenia ?) during 25 years, then he recovered!

recently killed with his wife in a car crash.

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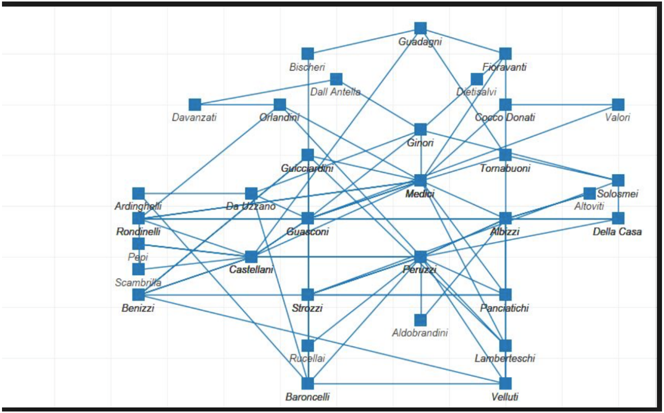

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- kind of problems: in Economics, understanding the form of network (organizations, nation states, web sites, scholarly publications,...)
- example: see social networks.
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Example 4: Price formation

- Abstract formalization of an exchange economy through an excess demand.
- An excess demand function is is the difference between supply and demand in good l.
- Question: existence of a zero ? comes from **Brouwer's theorem**
- And for computation (and proof ?): **Sperner Lemma**

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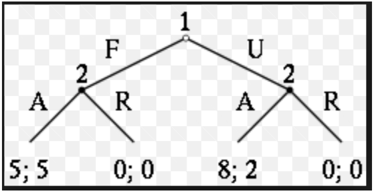
Example 5: Extensive form game.
Ultimatum game



Idea of subgame perfect equilibrium (improvement of Nash equilibrium)

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- What about if strategy spaces are not finite.
- Assume for example each strategy space is $[0, 1]$ each time, and each player plays one after the other.
- Assume the payoffs are continuous with respect to the path which is played.
- Is there a subgame perfect equilibrium ? method ?

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- Beyond technical questions: how to write a model for which something can be said ?
- Other question: which behaviour induces stability ?
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- 9*2 Hours
- Grade=an exam (approx 17 points)+a participation grade (approx 3 points)+potentially 2 additional points
- The exam=exercise or Questions related closely to the Lectures.
- Participation grade=attendance+small exams at the beginning of some lectures (additional points)+small oral presentation during the lectures.
- Typically, a student that makes a good oral presentation during a lecture, which attends the class, and which makes good small exams during the lectures will get 5 points, to be added to his final grade on 17.

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The concept of Continuity, compactness and convexity plays a very important role in this course.

Continuity of a function can be written using balls (closed or open), and the definition of balls uses the definition of metric spaces.

Definition: metric spaces

A metric space (E, d) is a set E together with a function $d : E \times E \to [0, +\infty[$, called a distance (or a metric) and satisfying:

- For every $(x, y) \in E \times E$, $d(x, y) = 0$ if and only if $x = y$;
- For every $(x, y) \in E \times E$, $d(x, y) = d(y, x)$;
- For every $(x, y, z) \in E \times E \times E$, $d(x, y) \leq d(x, z) + d(z, y)$;

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- For every $x \in E$, For every $t \in \mathbb{R}$, $\|tx\| = |t|\|x\|$;
- For every $(x, y) \in E \times E$, $\|x + y\| \leq \|x\| + \|y\|$.

Proposition

If $(E, \|.\|)$ is a norm space, then if we define $d(x, y) = \|x - y\|$ for every $(x, y) \in E \times E$, then (E, d) is a metric space.

- Example 0: discrete distance.
- Example 1: distance in a graph.
- Example 2: d_1, d_2, d_∞ in $\mathbf{R}^n$, $\|.\|_1$, $\|.\|_2$, $\|.\|_\infty$ in $\mathbf{R}^n$, or in space of real sequences.
- Example 3: d_∞ or $\|.\|_\infty$ on $\mathcal{B}(X, \mathbf{R})$, the set of bounded functions from X to $\mathbf{R}$.

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(Definition of Continuity of $f : X \to Y$ with balls):

Let (X, d) and (Y, δ) two metric spaces. Then
A function $f : X \to Y$ is continuous if and only if:

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Definition of compact subsets:

Let (E, d) a metric set. A subset $K \subset E$ is compact in E if for every familly $(U_i)_{i \in I}$ of open covering of K (i.e. $K \subset \cup_{i \in I} U_i$) if there is a finite subcovering (i.e. there is $J \subset I$, J finite, such that $K \subset \cup_{i \in J} U_i$).

Equivalent Definition of compact subsets:

Let (E, d) a metric set. A subset $K \subset E$ is compact in E if and only if: for every sequence (x_n) of K, there exists a subsequence $(x_{\phi(n)})$ which converges in K.

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Definition:

A metric space (X, d) has the fixed point property if: for every continuous function $f : X \to X$, there exists a fixed point $x \in X$ of X (which means $f(x) = x$).

Questions

$[0, 1]$ has the FPP ?
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$[0, 1] \cup [2, 3]$ has the FPP ?
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2. The Fixed-point property is preserved by homeomorphism

Definition:

Let (X, d) and (Y, δ) two metric spaces. A homeomorphism of X from Y if a function $f : X \to Y$ which is continuous, bijective, and such that $f^{-1} : Y \to X$ is continuous.

Theorem

Let (X, d) and (Y, δ) two metric spaces such that (X, d) has the Fixed Point property. If f is a a homeomorphism of X from Y then Y has the fixed point property.

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Let (X, d) a metric space and A a subspace of X. A continuous retraction from X to A is a continuous function $f : X \to A$ such that the restriction of f to A is the identity of A.

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Let (X, d) a metric space and A a subspace of X. If there exists f a continuous retraction from X to A and if (X, d) has the Fixed Point property, then (A, d) has the fixed point property.

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4. Convex compact subset in $\mathbf{R}^n$ have the Fixed-point property

Theorem (Brouwer theorem)

Convex compact subset in $\mathbf{R}^n$ have the Fixed-point property