Equilibria, fixed point, computation

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0. Presentation, syllabus

- Modelization in Micro, Macro, Network theory, game theory: what are we talking about ?
- A model in Economics, social science, ...: mathematical equations describing the evolution of an economical system (with agents).
- The agent can be a homo œconomicus (neo-classical paradigm: rational agent), or not (lack of rationality: cognitive limits, prudence, altruism, cognitive bias...)
- But his behaviour is influenced by some forces (happiness, social forces, altruism, ...) which can sometimes be contradictory.
- In a model, before studying the dynamic, interesting to study the stability of the system.
- Now, Equilibria comes on the scene!

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Let us say that an equilibrium is a description of a system (through some values of some variables) which does not move despite some forces.
Example:

But in Physics, there are very stable principle (Nature minimizes energy...)
And in Economics or Social sciences ?

Example 1: In game theory, Nash equilibrium.

Who is Nash ?

John Nash (American mathematician 1928-2015).
One of the greatest genius of mathematics according to some mathematicians. Won Nobel prize in Economics and Abel medal (like Nobel prize in mathematics).
Was ill (schizophrenia ?) during 25 years, then he recovered!
recently killed with his wife in a car crash.
See the film "A Beautiful Mind" by Ron Howard (Russel Crowe is Nash).

Example 1: In game theory, Nash equilibrium.

- definition of a normal form game G.
- definition of a Nash equilibrium x of G.
- Definition of the Best-replies.

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Three equivalent charatzerization of Nash equilibria

- 1) Fixed point of Best-replies $x \in M(x)$.
- 2) Fixed point of a well chosen function (original approach of Nash) $x = f(x)$.
- 3) Maximal element of some other multivalued function: $P(x) = \emptyset$.
- 4) zero of some well chosen function: $g(x) = 0$, where $g(x) = f(x) - x$.
- Then Questions: structural properties on the previous object that guarantees existence of...fixed-points, maximal element or zero of a function.

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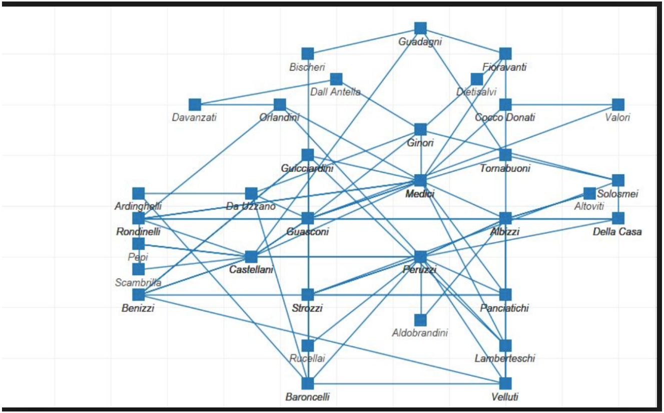
Example 3: Networks.



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- Theory of network is growing very fast!
- kind of problems: in Economics, understanding the form of network (organizations, nation states, web sites, scholarly publications,...)
- Basic notion of equilibrium: Pairwise stability notion.
- Formal defintion of a weighted network: N agents, links in $[0, 1]$.
- Formal definition of preferences of agent on the set of networks: utility function.
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Example 4: Price formation

- Abstract formalization of an exchange economy through an excess demand.
- Consider L goods; price of good l is $p_l \geq 0$.
- A vector of prices is $p = (p_1, ..., p_l)$ and is normalized $(\sum_l p_l^2 = 1)$
- Call S_+^{L-1} the set of price vectors.
- An excess demand function is $Z : S_+^{L-1} \to R^L$ where $Z(p) = (Z_1(p), ..., Z_L(p)$ and $Z_l(p)$ is the difference between supply and demand in good l. and satisfies:
- 1) Z continuous.
- 2) $p.Z(P) = 0$ for every p (Walras Law).
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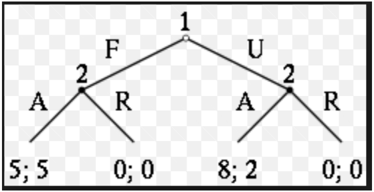
Example 5: Extensive form game.
Ultimatum game



Idea of subgame perfect equilibrium (improvement of Nash equilibrium)

Example 5: Extensive form game.

- What about if strategy spaces are not finite.
- Assume for example each strategy space is $[0, 1]$ each time, and each player plays one ofter the other.
- Assume the payoffs are continuous with respect to the path which is played.
- Is there a subgame perfect equilibrium ? method ?

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- Beyond technical questions: how to write a model for which something can be said ?
- Other question: which behaviour induces stability ?
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- Aim: give a tool to be able to say if an equation $f(x) = 0$ has at least one solution.
- Work for functions from a subset of $\mathbf{R}^n$ to $\mathbf{R}^n$ (can be generalized).
- i.e. as many equations as variables.
- We will associate to f an integer $\deg(f)$, call the degree of f, which is non zero when $f(x) = 0$ has a solution.

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1. Topological degree b) The bounded case ii) Homotopy

H

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Let f and g two continuous functions from $\overline{\Omega}$ to $\mathbf{R}^n$. A continuous homotopy between f and g is a mapping....

One says that the Homotopy has no zero on the boundary if ...

Remark1: If there is a Homotopy between f and g which has no zero on the boundary, then f and g have no zeros on the boundaries.

Remark2: There always exist a Homotopy between f and g, but there may not exist a homotopy which has no zero on the boundary.

- Let Ω open **bounded** subset of $\mathbf{R}^n$ (case A) described before)
- $\overline{\Omega}$ denotes the closure of open subset of Ω. Thus, $\overline{\Omega}$ is compact.
- $\partial\Omega$ denotes the boundary of Ω.
- Recall if $f : \Omega \to \mathbf{R}^n$ is C^1, then the Jacobian of f at $x \in \Omega$ is ...
- Recall if $f : \Omega \to \mathbf{R}^n$ is C^1, then the first order development at x is ....
- **Bolzano Weierstrass:** If (x_n) is a bounded sequence of a finite dimensional space, there exists a convergent subsequence.
- **Compacity in $\mathbf{R}^n$** $K \subset \mathbf{R}^n$ is compact if and only if it is bounded and closed, if and only if every sequence of K has a subsequence which converges in K.

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Let f and g two continuous functions from $\overline{\Omega}$ to $\mathbf{R}^n$. A continuous homotopy between f and g is a mapping....
One says that the Homotopy has no zero on the boundary if ...

Remark1: If there is a Homotopy between f and g which has no zero on the boundary, then f and g have no zeros on the boundaries.
Remark2: There always exist a Homotopy between f and g, but there may not exist a homotopy which has no zero on the boundary.

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To every f continuous function from $\overline{\Omega}$ to $\mathbf{R}^n$ (where Ω open and bounded in $\mathbb{R}^n$) with no zero on the boundary (i.e.$\forall x \in \partial\Omega, f(x) \neq 0$), we can associate its **topological degree**, denoted $\deg(f) \in \mathbb{Z}$ such that:

1) **Identity.** $\deg(id)$=1 if $0 \in \Omega$.

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4) **Additivity.** Let Ω_1 and Ω_2 two open disjoint subsets of Ω and $f : \Omega \to \mathbb{R}$ a continuous function such that $f^{-1}(0)$ included in $\Omega_1 \cup \Omega_2$. Then $\deg(f)$=$\deg(f_{|\Omega_1})$+$\deg(f_{|\Omega_2})$.

5) **Unvariance.** $\deg(f)$ =$\deg(g)$ for every $f : \overline{\Omega} \to \mathbb{R}^n$ and $g : \overline{\Omega} \to \mathbb{R}^n$ with no zero on the boundary and such that $\|f - g\|_\infty < d(0, f(\partial\Omega))$.

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We now allow Ω to be non bounded, but we impose conditions on the mappings and on the homotopy ("Compactly rooted") so that the possible sets of roots that will appear are compact!

Compactly rooted mapping

The continuous function f from $\overline{\Omega}$ to $\mathbf{R}^n$ is said to be compactly rooted if $f^{-1}(0)$ is a compact subset of $\mathbf{R}^n$.

Homotopy

Let f and g two continuous functions from $\overline{\Omega}$ to $\mathbf{R}^n$. A continuous homotopy between f and g is said to be compactly rooted if $H^{-1}(0)$ is a compact subset of $[0, 1] \times \mathbf{R}^n$.

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Proposition about Regularity in the unbounded case

Let f a C^1 functions from $\overline{\Omega}$ to $\mathbf{R}^n$ which is regular and has no zero on the boundary of Ω. Then the set of zero of f is finite.

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To every f continuous and **compactly rooted** function from $\overline{\Omega}$ to $\mathbf{R}^n$ (where Ω open in $\mathbb{R}^n$) with no zero on the boundary (i.e.$\forall x \in \partial\Omega, f(x) \neq 0$), we can associate its **topological degree**, denoted $\deg(f) \in \mathbb{Z}$ such that:

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Exercise (in Last Year exam).
Prove that the following system admits at least one solution $(x, y) \in \mathbf{R}^2$:

$$x + y = cos(yx)$$

and

$$x - y = cos(x).$$

Please justify precisely each step of your method. All computations should be explicited.