On the continuous representation of quasi-concave mappings by their upper level sets.

Philippe Bich [a]

[a] *Paris School of Economics, Centre d'Economie de la Sorbonne UMR 8174, Université Paris I Panthéon Sorbonne, Paris, France. Tel: 01 44 07 83 14. Fax: 01 44 07 83 01*

Abstract

We provide a continuous representation of quasi-concave mappings by their upper level sets. A possible motivation is the extension to quasi-concave mappings of a result by Ulam and Hyers, which states that every approximately convex mapping can be approximated by a convex mapping.

Key words: quasi-concave, upper level set

1 Introduction

Since a seminal paper of Hyers and Ulam (1), several papers have studied the problem of the stability of functional inequalities. In particular, the first issue given (and solved) by Hyers and Ulam was roughly the following: is it possible to approximate approximately convex mappings by convex mappings ?

Now, an important extension of convexity, especially for mathematical economics or game theory, is quasi-concavity. In the following, let X be a convex subset of some vector space.

Definition 1.1 *A mapping $f : X \to \mathbb{R}$ is quasi-concave if the following property is true:*

$$\forall (x, y) \in X \times X,\ \forall \lambda \in [0, 1],\ f(\lambda.x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}.$$

Email address: bich@univ-paris1.fr (Philippe Bich).

Preprint submitted to Elsevier *16 October 2008*

Equivalently, $f : X \to \mathbb{R}$ is quasi-concave if its upper level sets $C(\lambda) = \{x \in X,\ f(x) \geq \lambda\}$ are convex subsets of X for every $\lambda \in \mathbb{R}$. Clearly, concave and convex mappings are quasi-concave. Besides, non-increasing or non-decreasing mappings from $\mathbb{R}$ to $\mathbb{R}$ are quasi-concave.

In order to raise the issue of the stability of the quasi-concavity property, one defines the notion of approximately quasi-concave mapping as follows:

Definition 1.2 *For every $\epsilon > 0$, the mapping $f : X \to \mathbb{R}$ is $\epsilon-$quasi-concave if the following property is true:*

$$\forall (x, y) \in X \times X,\ \forall \lambda \in [0, 1],\ f(\lambda.x + (1 - \lambda)y) \geq \min\{f(x), f(y)\} - \epsilon.$$

A natural question, in the vein of Hyers and Ulam's stability result, is to know whether it possible to approximate an ϵ-quasi-concave mapping $f : X \to \mathbb{R}$ by a quasi-concave mapping. Such question could have applications in game theory or in mathematical economics, but it also has an interest for its own sake.

A possible answer to this question could be to build the quasi-concave envelop $\tilde{f}$ of f as follows: first, associate to the $\epsilon-$quasi-concave mapping f its upper level sets $C^f(\lambda) = \{x \in X,\ f(x) \geq \lambda\}$. Then convexify the upper level sets by defining their convex hulls co$C^f(\lambda)$. Last, try to find a quasi-concave mapping $\tilde{f}$ whose upper level sets are the co$C(\lambda)$.

A first objective of the present paper is to clarify the previous construction. [1] In particular, one could hope that the mapping $\tilde{f}$ is a good approximation of f if ϵ is small enough. In view of the construction of $\tilde{f}$, a question which naturally arises is the continuity of the representation of each mapping $f : X \to \mathbb{R}$ by its upper level sets. This is the main purpose of this paper, and this study is given in Section 2. In particular, it requires the definition of a good topology on the set of upper level sets. Besides, the stability problem presented above suggests to study the continuity properties of the convex hull operator in the set of upper level sets, which is done in Section 3. The stability of the quasi-concave property is given in Section 4, as a immediate byproduct of the previous sections.

[1] Other constructions of the quasi-concave envelop could be given. But we think the construction here given, which is very natural, raises interesting questions we would like to answer in this paper.

2 Continuous representation of a quasi-concave mapping by its upper level sets

The purpose of this section is to define the continuous representation of quasi-concave mappings by their upper level sets. In the following subsection, we define the basic object that represents quasi-concave mappings, and define a generalized metric structure on this object.

2.1 Definition of u.l.s. multivalued mappings

Definition 2.1 *Let X be a vector space. A multivalued mapping C from $\mathbb{R}$ to X is said to be a u.l.s. (upper level sets) multivalued mapping if the following properties are satisfied:*

i) **Monotony** *For every $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$, $\lambda \leq \mu$ implies $C(\mu) \subset C(\lambda)$.*

ii) **Left Continuity** *For every $\lambda \in \mathbb{R}$, one has $\bigcap_{n \in \mathbb{N}^*} C(\lambda - \frac{1}{n}) = C(\lambda)$.*

iii) **Limits at $+\infty$ and $-\infty$** *One has $\bigcup_{\lambda \in \mathbb{R}} C(\lambda) = X$ and $\bigcap_{\lambda \in \mathbb{R}} C(\lambda) = \emptyset$.*

Throughout this paper, $\mathcal{C}$ denotes the set of all multivalued mappings from $\mathbb{R}$ to X satisfying the Assumptions i) and iii) above, and $\mathcal{C}^{u.l.s.}$ the set of all u.l.s. multivalued mappings from $\mathbb{R}$ to X.

In the following proposition 2.2, whose proof is given in the appendix, we define a generalized metric structure on $\mathcal{C}^{u.l.s.}$. Recall that for every set E, a mapping $\delta : E \times E \to \mathbb{R}^+ \cup \{+\infty\}$ is a generalized pseudo metric if these three conditions are true:

(i) for every $x \in E$, $d(x, x) = 0$.

(ii) for every $(x, y) \in E \times E$, $d(x, y) = d(y, x)$.

(iii) for every $(x, y, z) \in E^3$, one has $d(x, z) \leq d(x, y) + d(y, z)$.

The generalized pseudo metric d is a generalized metric if in addition, for every $(x, y) \in E^2$, $d(x, y) = 0$ implies $x = y$.

In the following proposition, by convention, the infimum of an empty set is $+\infty$.

Proposition 2.2 *The mapping* $\delta : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ *defined for every* $(C, C') \in \mathcal{C} \times \mathcal{C}$ *by*

$$\delta(C, C') = \sup_{\lambda \in \mathbb{R}} \{\max\{\inf\{a \geq 0 \mid C(\lambda) \subset C'(\lambda - a)\}, \inf\{a \geq 0 \mid C'(\lambda) \subset C(\lambda - a)\}\}\}$$

is a generalized pseudo-metric, and its restriction to $\mathcal{C}^{u.l.s.} \times \mathcal{C}^{u.l.s.}$ *is a generalized metric.*

Remark 2.3 *It is easy to build a metric from the generalized metric* δ*, for example by defining* $d : \mathcal{C}^{u.l.s.} \times \mathcal{C}^{u.l.s.} \rightarrow \mathbb{R}^+$ *by*

$$\forall (C, C') \in \mathcal{C}^{u.l.s.} \times \mathcal{C}^{u.l.s.}, \; d(C, C') = \min\{1, \delta(C, C')\}.$$

2.2 Definition of a topology on C

We now define a topology on $\mathcal{C}$, thanks to the generalized pseudo metric δ defined above.

Definition 2.4 *Let* $\mathcal{T}$ *be the topology on* $\mathcal{C}$ *induced by the previous generalized pseudo metric* δ*. More precisely, this topology is generated by the collection of open sets* $B(x, r) = \{y \in \mathcal{C}, \delta(x, y) < r\}$ *where* $x \in \mathcal{C}$ *and* $r > 0$*.*

Throughout this paper, $\mathcal{C}$ is equipped with the topology $\mathcal{T}$, and the topological space such defined will be denoted $(\mathcal{C}, \delta)$. Notice that from the properties satisfied by the generalized pseudo-metric δ, the collection of sets $B(x, r)$ is a basis of $\mathcal{T}$: it means that every open set for $\mathcal{T}$ is a union of some $B(x, r)$.

Since we aim at representing continuously quasi-concave mappings from X to $\mathbb{R}$ by elements of $\mathcal{C}^{u.l.s.}$, we now have to define a topology on $\mathcal{F}$, the set of all mappings from X to $\mathbb{R}$.

Definition 2.5 *For every* $(f, g) \in \mathcal{F}$*, define* $\|f - g\|_\infty = \sup_{x \in X} \mid f(x) - g(x) \mid$*. Let* $\mathcal{U}$ *be the topology on* $\mathcal{F}$ *generated by the collection of open sets* $B(f, r) = \{g \in \mathcal{F}, \|f - g\|_\infty < r\}$*, where* $f \in \mathcal{F}$ *and* $r > 0$*.*

Remark that $\|.\|_\infty$ is not a norm. Nevertheless, $(\mathcal{F}, \|.\|_\infty)$ will denote the set $\mathcal{F}$ equipped with the topology $\mathcal{U}$. Notice that the collection of sets $B(x, r)$ is a basis of $\mathcal{U}$.

2.3 The continuous representation

Let $\mathcal{F}^q$ be the set of all quasi-concave mappings in $\mathcal{F}$, and for every $\epsilon > 0$, let $\mathcal{F}^q_\epsilon$ be the set of all $\epsilon-$quasi-concave mappings in $\mathcal{F}$. The sets corresponding to $\mathcal{F}^q$ and $\mathcal{F}^q_\epsilon$ through our representation will be $\mathcal{C}^{conv}$ and $\mathcal{C}^{conv}_\epsilon$: the set $\mathcal{C}^{conv}$ is the set of all multivalued mappings of $\mathcal{C}^{u.l.s.}$ with convex (possibly empty) values; the set $\mathcal{C}^{conv}_\epsilon$ is defined as follows:

Definition 2.6 *The set $\mathcal{C}^{conv}_\epsilon$ is the set of all multivalued mappings of $\mathcal{C}^{u.l.s.}$ such that for every $\lambda \in \mathbb{R}$, one has $coC(\lambda) \subset C(\lambda - \epsilon)$.*

We will see in the next section the interpretation of this last condition in terms of distance, which is roughly that the multivalued mapping C and its convex hull co$C$ are not too far. Throughout this paper, all the subsets of $\mathcal{F}$ and of $\mathcal{C}$ are equipped with the topologies induced by $\mathcal{U}$ and $\mathcal{T}$.

We now state our main representation theorem: it explicitly describes the continuous representation from $\mathcal{F}$ to $\mathcal{C}^{u.l.s.}$.

Theorem 2.7 *i) The mapping $\Phi : (\mathcal{F}, \|.\|_\infty) \to (\mathcal{C}^{u.l.s.}, \delta)$ defined by*

$$\forall f \in \mathcal{F}, \forall \lambda \in \mathbb{R},\ \Phi(f)(\lambda) = C^f(\lambda) := \{x \in X,\ f(x) \geq \lambda\}$$

is an isometric isomorphism, i.e. Φ is bijective and for every $(f, g) \in \mathcal{F} \times \mathcal{F}$, one has $\|f - g\|_\infty = \delta(\Phi(f), \Phi(g))$. Besides, $\Psi = \Phi^{-1}$ is defined by

$$\forall C \in \mathcal{C}^{u.l.s.},\ \Psi(C)(x) = \sup\{\lambda \in \mathbb{R} \mid x \in C(\lambda)\}.$$

ii) Φ is an isometric isomorphism from $\mathcal{F}^q$ to $\mathcal{C}^{conv}$, and also from $\mathcal{F}^q_\epsilon$ to $\mathcal{C}^{conv}_\epsilon$.

Proof. See the appendix.

3 The Convex hull operator

In this section, one defines the convex hull operator co from the set of all multivalued mapping from $\mathbb{R}$ to X to itself as follows: for every multivalued mapping C from $\mathbb{R}$ to X and for every $\lambda \in \mathbb{R}$, $(\text{co}C)(\lambda)$ is the convex hull of $C(\lambda)$.

We have seen in the introduction that a natural method to build the quasi-concave envelop of a mapping is to take the convex hull co$\Phi(f)$ of the upper level sets of f, then to define a new quasi-concave mapping $\Phi^{-1}(\text{co}\Phi(f))$, if it is possible. A problem is that the multivalued mapping co$\Phi(f)$ may not be in $\mathcal{C}^{u.l.s.}$, so that $\Phi^{-1}(\text{co}\Phi(f))$ could be not well defined. The following example provides a u.l.s. multivalued mapping C from $\mathbb{R}$ to $\mathbb{R}$ whose convex hull is not u.l.s.

Example 1 Let C be the multivalued mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$C(x) = [x, -x] \cup ([0, 1[\cap \mathbb{Q}) \text{ if x} \leq 0$$
$$C(x) = ([0, 1 - x[\cup[x, 1[) \cap \mathbb{Q} \text{ if x} \in [0, 1[$$
$$C(1) = \{0\}$$
$$C(x) = \emptyset \text{ if x} > 1$$

It is easy to check that $C \in \mathcal{C}^{u.l.s.}$. Yet, the multivalued mapping coC is defined by

$$coC(x) = [x, -x] \text{ if x} \leq -1$$
$$coC(x) = [x, 1[\text{ if x} \in]-1, 0]$$
$$coC(x) = [0, 1[\text{ if x} \in [0, 1[$$
$$coC(1) = \{0\}$$
$$coC(x) = \emptyset \text{ if x} > 1$$

and clearly, coC does not satisfy Property ii) of Definition 2.1 at $x = 1$ (left continuity), so is not u.l.s.

Example 2 Let f be the mapping from $\mathbb{R}^2$ to $\mathbb{R}$ defined as follows: for every $(x, y) \in \mathbb{R}^2$, $f(x, y)$ is the opposite of the Euclidean distance between (x, y) and $K = \{(0, 0)\} \cup \{(x, \frac{1}{x}) \in \mathbb{R}^2, x < 0\}$. The mapping f is continuous, $\Phi(f)(0) = K$ and $co\Phi(f)(0) = coK$ is not closed. Now, for every integer $n > 0$, one has $\overline{coK} \subset co\Phi(f)(-\frac{1}{n})$, so that $\bigcap_{n\in\mathbb{N}^*} co\Phi(f)(-\frac{1}{n}) \neq co\Phi(f)(0)$, and so the multivalued function $co\Phi(f)$ is not u.l.s.

So, the first task to avoid this problem is to extend $\Psi = \Phi^{-1}$ on a larger set. We prove that Ψ is, as a matter of fact, an isometry on all the set $\mathcal{C}$:

Proposition 3.1

$$\forall C \in \mathcal{C},\ \Psi(C)(x) = \sup\{\lambda \in \mathbb{R} \mid x \in C(\lambda)\}$$

is well defined, and Ψ is an isometry. More precisely, for every $(C, C') \in \mathcal{C} \times \mathcal{C}$, one has $\|\Psi(C) - \Psi(C')\|_\infty = \delta(C, C')$.

Proof. See the appendix.

Unfortunately, there could remain some multivalued mapping $C \in \mathcal{C}^{u.l.s.}$ such that co$C$ is not in $\mathcal{C}$, as shown in the following example:

Example 3 Let C be the multivalued mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$C(x) =]-\infty, -x] \cup [x, +\infty[\text{ if x} \geq 0$$
$$C(x) = \mathbb{R} \text{ if x} \leq 0$$

It is clear that $C \in \mathcal{C}^{u.l.s.}$, *but* $coC \notin \mathcal{C}$, *because* $\bigcap_{x\in\mathbb{R}} coC(x) = \mathbb{R}$.

So, the second task is to find a subset $\mathcal{C}'$ of $\mathcal{C}$ such that for every $C \in \mathcal{C}'$, $coC \in \mathcal{C}$. This is the aim of the following definition:

Definition 3.2 *Define* $\mathcal{C}'$ *the set of multivalued mappings of* $\mathcal{C}$ *such that*

$$\bigcap_{\lambda\in\mathbb{R}} coC(\lambda) = \emptyset.$$

It is clear from the previous definition that $co(\mathcal{C}') \subset \mathcal{C}$. Remark that for every bounded mapping $f : X \to \mathbb{R}$, $\Phi(f) \in \mathcal{C}'$, because $\Phi(f)(n) = \emptyset$ for $n \in \mathbb{N}$ large enough.

The following proposition relates this new set $\mathcal{C}'$ to $\mathcal{C}^{conv}_{\epsilon}$:

Proposition 3.3 *For every* $\epsilon > 0$, $\mathcal{C}^{conv}_{\epsilon} \subset C'$.

Proof of Proposition 3.3 For every $\epsilon > 0$, let $C \in \mathcal{C}^{conv}_{\epsilon}$. One has $\bigcap_{\lambda\in\mathbb{R}} \mathrm{co}C(\lambda) \subset \bigcap_{\lambda\in\mathbb{R}} C(\lambda - \epsilon)$ (from the definition of $\mathcal{C}^{conv}_{\epsilon}$). This last set is also $\bigcap_{\lambda\in\mathbb{R}} C(\lambda) = \emptyset$ from Property iii) of the definition of $C^{u.l.s.}$.

To finish, notice that each point of $\mathcal{C}^{conv}_{\epsilon}$ is almost stabilized by the convex hull operator:

Proposition 3.4 *For every* $C \in \mathcal{C}^{conv}_{\epsilon}$, $\delta(co(C), C) \leq \epsilon$.

Proof. This is clear from the definition of $\mathcal{C}^{conv}_{\epsilon}$.

4 Stability of quasi-concavity

To illustrate the utility of the previous results, one proves the stability of the quasi-concave property. Let $f \in \mathcal{F}^q_{\epsilon}$ an $\epsilon-$quasi-concave mapping from X to $\mathbb{R}$. By Theorem 2.7, $\Phi(f) \in \mathcal{C}^{conv}_{\epsilon}$, so $\Phi(f) \in \mathcal{C}'$ from Proposition 3.3. Since $co(\mathcal{C}') \subset \mathcal{C}$, one has $co(\Phi(f)) \in \mathcal{C}$. Hence, from Proposition 3.1, $\tilde{f} := \Psi(co(\Phi(f)))$ is well defined. Besides, since $\Psi(\Phi(f)) = f$, one has $\|\tilde{f} - f\|_{\infty} = \|\Psi(co(\Phi(f))) - \Psi(\Phi(f))\|_{\infty}$. Now, from Proposition 3.1, Ψ is an isometry on $\mathcal{C}$; thus, one has $\|\Psi(co(\Phi(f))) - \Psi(\Phi(f))\|_{\infty} = \delta(co(\Phi(f)), \Phi(f)) \leq \epsilon$ from Proposition 3.4 applied to $C = \Phi(f) \in \mathcal{C}^{conv}_{\epsilon}$.

Thus, one obtains a result similar to Hyers-Ulam's one for quasi-concave mappings:

Proposition 4.1 *For every $\epsilon-$quasi-concave mapping $f : X \to \mathbb{R}$ there exists $\tilde{f} : X \to \mathbb{R}$ a quasi-concave mapping such that $\|\tilde{f} - f\|_\infty \leq \epsilon$.*

5 Appendix

Proof of Proposition 2.2. First notice that by definition, $\delta(C, C')$ is positive for every $(C, C') \in \mathcal{C} \times \mathcal{C}$, and may be infinite. Besides, $\delta(C, C) = 0$ for every $C \in \mathcal{C}$, and δ is clearly symmetric. To prove the triangular inequality, let $(C, C', C'') \in \mathcal{C}^3$ such that $\delta(C, C') < +\infty$ and $\delta(C, C') < +\infty$ (if one of these two inequalities are not true, then the inequality $\delta(C, C'') \leq \delta(C, C') + \delta(C', C'')$ is clearly true). First, one has

$$\forall \lambda \in \mathbb{R}, \forall \epsilon > 0, C(\lambda) \subset C'(\lambda - \delta(C, C') - \epsilon). \tag{1}$$

Indeed, from the definition of $\delta(C, C')$, for every $\lambda \in \mathbb{R}$ and for every $\epsilon > 0$, there exists $a \in \mathbb{R}$ such that $\delta(C, C') \geq a$ and such that one has $C(\lambda) \subset C'(\lambda - a - \epsilon)$ and $C'(\lambda) \subset C(\lambda - a - \epsilon)$. Since $\lambda - a - \epsilon \geq \lambda - \delta(C, C') - \epsilon$, and from the monotony assumption satisfied by C and C', one obtains Equation 1. Second, one can apply Equation 1 to C' and C'', replacing λ by $\lambda - \delta(C, C') - \epsilon$. One obtains

$$\forall \lambda \in \mathbb{R}, \forall \epsilon > 0, C'(\lambda - \delta(C, C') - \epsilon) \subset C''(\lambda - \delta(C, C') - \delta(C', C'') - 2\epsilon). \tag{2}$$

Combination of Equations 1 and 2 gives, for every $\lambda \in \mathbb{R}$ and every $\epsilon > 0$, $C(\lambda) \subset C''(\lambda - \delta(C, C') - \delta(C', C'') - 2\epsilon)$ and similarly, by symmetry, $C''(\lambda) \subset C(\lambda - \delta(C, C') - \delta(C', C'') - 2\epsilon)$. Thus, from the definition of $\delta(C, C'')$, for every $\epsilon > 0$ one has $\delta(C, C'') \leq \delta(C, C') + \delta(C', C'') + 2\epsilon$ which entails the triangle inequality.

Thus, δ is an generalized pseudo metric, that is to say a pseudo metric that can take infinite values. Besides, if $(C, C') \in \mathcal{C}^{u.l.s.} \times \mathcal{C}^{u.l.s.}$ and if $\delta(C, C') = 0$ then clearly, for every $\lambda \in \mathbb{R}$, $C(\lambda) \subset \bigcap_{n \in \mathbb{N}} C'(\lambda - \frac{1}{n})$. But from Property ii) of Definition 2.1, this last set is equal to $C'(\lambda)$; similarly, one obtains, for every $\lambda \in \mathbb{R}$, $C'(\lambda) \subset C(\lambda)$, and finally $C = C'$. Thus, the restriction of δ to $\mathcal{C}^{u.l.s.} \times \mathcal{C}^{u.l.s.}$ is a generalized metric.

Proof of Theorem 2.7

Statement i. Let $f : X \to \mathbb{R}$. First prove that $\Phi(f) = C^f \in \mathcal{C}^{u.l.s.}$. We have to check the three conditions i), ii) and iii) of Definition 2.1. To prove i), let $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$ such that $\lambda \leq \mu$. If $x \in C^f(\mu)$ then $f(x) \geq \mu \geq \lambda$, thus $x \in C^f(\lambda)$. To prove ii), take $\lambda \in \mathbb{R}$ and $x \in \bigcap_{n\in\mathbb{N}} C^f(\lambda - \dfrac{1}{n})$. Thus $f(x) \geq \lambda - \frac{1}{n}$ for every integer n, and passing to the limit one obtains $f(x) \geq \lambda$, or equivalently $x \in C^f(\lambda)$. Last, to prove iii), just notice that for every $x \in X$, one has $x \in C^f(f(x))$ and $x \notin C^f(f(x) + 1)$.

Second, let us prove that for every $C \in \mathcal{C}^{u.l.s.}$, $\Psi(C)$ is well defined. Let $C \in \mathcal{C}^{u.l.s.}$. For every $x \in X$, define $E(x) = \{\lambda \in \mathbb{R}, x \in C(\lambda)\}$. Since $\bigcup_{\lambda\in\mathbb{R}} C(\lambda) = \mathbb{R}$ (Property iii) of Definition 2.1), the set $E(x)$ is non empty. Since $\bigcap_{\lambda\in\mathbb{R}} C(\lambda) = \emptyset$ (Property iii) of Definition 2.1) and from Property i) of Definition 2.1, the set $E(x)$ is majorized, which finally proves that Ψ is well defined.

Third, we prove that Ψ is a right inverse of Φ, i.e. that for every $C \in \mathcal{C}^{u.l.s.}$, one has $\Phi(\Psi(C)) = C$. This is equivalent to prove that for every $\mu \in \mathbb{R}$, one has $\{x \in X, \Psi(C)(x) \geq \mu\} = C(\mu)$. To prove $C(\mu) \subset \{x \in X, \Psi(C)(x) \geq \mu\}$, take $x \in C(\mu)$, and notice that by definition of $\Psi(C)$, one has $\Psi(C)(x) \geq \mu$. To prove $\{x \in X, \Psi(C)(x) \geq \mu\} \subset C(\mu)$, suppose $\Psi(C)(x) \geq \mu$. By definition of $\Psi(C)$ and by Property i) of Definition 2.1, one has $x \in \bigcap_{n\in\mathbb{N}^*} C(\mu - \dfrac{1}{n})$ which is equal to $C(\mu)$ from Property ii) of Definition 2.1. This finally proves $\Phi(\Psi(C)) = C$; in particular, Φ is onto.

Now, prove that Ψ is a left inverse of Φ, i.e. that $\Psi(\Phi(g)) = g$ for every $g \in \mathcal{F}$. We have to prove the equality $g(x) = \sup\{\lambda \in \mathbb{R} \mid x \in \Phi(g)(\lambda)\}$ for every $g \in \mathcal{F}$. First, the inequality $g(x) \geq \sup\{\lambda \in \mathbb{R} \mid x \in \Phi(g)(\lambda)\}$ is clear, because from the definition of $\Phi(g)(\lambda)$, one has $g(x) \geq \lambda$ for every $x \in \Phi(g)(\lambda)$. Second, the inequality $g(x) \leq \sup\{\lambda \in \mathbb{R} \mid x \in \Phi(g)(\lambda)\}$ is due to $x \in \Phi(g)(g(x))$.

Thus, we have proved that Φ and Ψ are well defined and that $\Psi = \Phi^{-1}$. To finish the proof of Theorem 2.7, one has to prove that for every $(f, g) \in \mathcal{F} \times \mathcal{F}$, $\|f - g\|_\infty = \delta(\Phi(f), \Phi(g))$. Let $(f, g) \in \mathcal{F} \times \mathcal{F}$. Define $\epsilon := \|f - g\|_\infty \in \mathbb{R} \cup \{+\infty\}$

First case: ϵ finite. We want to prove that $\delta(\Phi(g), \Phi(f)) = \epsilon$. First, prove the inequality $\delta(\Phi(g), \Phi(f)) \leq \epsilon$. This would be a consequence of the fact that for every $\lambda \in \mathbb{R}$, $\Phi(g)(\lambda) \subset \Phi(f)(\lambda - \epsilon)$ and $\Phi(f)(\lambda) \subset \Phi(g)(\lambda - \epsilon)$. To prove the first inclusion, let $x \in \Phi(g)(\lambda)$, which means $g(x) \geq \lambda$. From $\|f - g\|_\infty = \epsilon$, one deduces that $f(x) \geq \lambda - \epsilon$, thus $x \in \Phi(f)(\lambda - \epsilon)$. The second inclusion is proved similarly. Now, prove the converse inequality $\delta(\Phi(g), \Phi(f)) \geq \epsilon$. If this inequality is false, then there exists $a > 0$ such that $\delta(\Phi(g), \Phi(f)) < \epsilon - a$. From the definition of Ψ, for every $x \in X$ there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to $\Psi(\Phi(f))(x) = f(x)$ such that $x \in \Phi(f)(\lambda_n)$ for every integer n. From $\delta(\Phi(f), \Phi(g)) < \epsilon - a$, the condition $x \in \Phi(f)(\lambda_n)$ implies $x \in \Phi(g)(\lambda_n - \epsilon + a)$, which is equivalent by definition to $g(x) \geq \lambda_n - \epsilon + a$. Passing to the limit, one obtains $g(x) \geq f(x) - \epsilon + a$, and similarly, one proves $f(x) \geq g(x) - \epsilon + a$. Thus, $\|f(x) - g(x)\|_\infty < \epsilon$, a contradiction with the definition of ϵ.

Second case: $\epsilon = +\infty$. We want to prove that $\delta(\Phi(g), \Phi(f)) = +\infty$. The proof is similar to the end of the proof of the first case: suppose that $\delta(\Phi(g), \Phi(f)) < A$ for some $A \in \mathbb{R}$. From the definition of Ψ, for every $x \in X$, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to $\Psi(\Phi(f))(x) = f(x)$ such that $x \in \Phi(f)(\lambda_n)$ for every integer n. From $\delta(\Phi(f), \Phi(g)) < A$, the condition $x \in \Phi(f)(\lambda_n)$ implies $x \in \Phi(g)(\lambda_n - A)$, or equivalently $g(x) \geq \lambda_n - A$. Passing to the limit, one obtains $g(x) \geq f(x) - A$, and similarly, one proves $f(x) \geq g(x) - A$. Thus, $\|f(x) - g(x)\|_\infty \leq A$, a contradiction with the definition of ϵ.

Statement ii.) It is well-known that f is quasi-concave if and only if its upper level sets are convex; thus, Φ is an isometric isomorphism from $\mathcal{F}^q$ to $\mathcal{C}^{conv}$. Now, let us prove that f is ϵ-quasi-concave if and only $\Phi(f) \in \mathcal{C}_\epsilon^{conv}$. First suppose that f is ϵ-quasi-concave. Let $\lambda \in \mathbb{R}$ and $x \in$co$C^f(\lambda)$. Thus, there exists an integer n and x_1, ...,x_n in $C^f(\lambda)$ such that $x = \sum_{i=1}^n \mu_i x_i$, with $\mu_i \geq 0$ for every $i = 1, ..., n$ and $\sum_{i=1}^n \mu_i = 1$. From the definition of $C^f(\lambda)$, for every $i = 1, ..., n$ one has $f(x_i) \geq \lambda$; since f is $\epsilon-$quasi-concave, one obtains $f(x) \geq \lambda - \epsilon$, or equivalently $x \in C^f(\lambda - \epsilon)$. We have proved that for every $\lambda \in \mathbb{R}$, one has co$C^f(\lambda) \subset C^f(\lambda - \epsilon)$, or equivalently, $\Phi(f) \in \mathcal{C}_\epsilon^{conv}$. Conversely, suppose co$C^f(\lambda) \subset C^f(\lambda - \epsilon)$ for every $\lambda \in \mathbb{R}$. Let $x = \sum_{i=1}^n \mu_i x_i$, with $\mu_i \geq 0$ for every $i = 1, ..., n$ and $\sum_{i=1}^n \mu_i = 1$. Define $\lambda = \min\{f(x_1), ..., f(x_n)\}$, so that one has $f(x_i) \geq \lambda$ for every $i = 1, ..., n$; equivalently, this means $x_i \in C^f(\lambda)$ for every $i = 1, ..., n$. Consequently, one has $x \in coC^f(\lambda)$. From co$C^f(\lambda) \subset C^f(\lambda - \epsilon)$, one finally obtains $x \in C^f(\lambda - \epsilon)$, which is equivalent to $f(x) \geq \min\{f(x_1), ..., f(x_n)\} - \epsilon$. This proves that f is $\epsilon-$quasi-concave. Finally, Φ is an isometric isomorphism from $\mathcal{F}_\epsilon^q$ to $\mathcal{C}_\epsilon^{conv} \cap \mathcal{C}^{u.l.s.}$.

Proof of Proposition 3.1 Notice that from the proof of Theorem 2.7, Statement i), second point, Ψ is well defined on $\mathcal{C}$, and not only on $\mathcal{C}^{u.l.s.}$. However, if Ψ is extended on all $\mathcal{C}$, then it is no longer true that Ψ is a right inverse of Φ (the proof of this property uses Property ii) of Definition 2.1, which can be false in $\mathcal{C}$.). Thus, the fact that for every $(f, g) \in \mathcal{F} \times \mathcal{F}$, $\|f - g\|_\infty = \delta(\Phi(f), \Phi(g))$ is of no help to prove that for every $(C, C') \in \mathcal{C} \times \mathcal{C}$, $\delta(C, C') = \|\Psi(C) - \Psi(C')\|_\infty$. We have to give a direct proof.

First prove that $\delta(C, C') \leq \|\Psi(C) - \Psi(C')\|_\infty$. By definition of $\delta(C, C')$, there exist two sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, where $(\mu_n)_{n\in\mathbb{N}}$ converges to $\delta(C, C')$, and such that for every integer n, $C(\lambda_n)$ is not contains in $C'(\lambda_n - \mu_n)$. It means that for every integer n, there exists $x_n \in X$ such that $x_n \in C(\lambda_n)$ and

$x_n \notin C'(\lambda_n - \mu_n)$. Thus, by definition, for every integer n one has $\Psi(C)(x_n) \geq \lambda_n$ and $\Psi(C')(x_n) \leq \lambda_n - \mu_n$. Thus, $\|\Psi(C) - \Psi(C')\|_\infty \geq \mu_n$, and one obtains at the limit $\|\Psi(C) - \Psi(C')\|_\infty \geq \delta(C, C')$. Notice that this proof works if $\delta(C, C') = +\infty$.

Now prove that $\delta(C, C') \geq \|\Psi(C) - \Psi(C')\|_\infty$. Suppose this last inequality is false, i.e. $\delta(C, C') < \|\Psi(C) - \Psi(C')\|_\infty$. Thus, there exists $x \in X$ such that for $\epsilon > 0$ small enough, one has $\Psi(C)(x) - \Psi(C')(x) > \delta(C, C') + \epsilon$ (switching C and C' is necessary). Thus, from the definition of $\Psi(C')(x)$, one has $x \notin C'(\Psi(C)(x) - \delta(C, C') - \epsilon)$. But from the definition of $\Psi(C)(x)$, one also have $x \in C(\Psi(C)(x) - \frac{\epsilon}{2})$. From the definition of $\delta(C, C')$, it implies $\delta(C, C') \geq (\Psi(C)(x) - \frac{\epsilon}{2}) - (\Psi(C)(x) - \delta(C, C') - \epsilon) = \delta(C, C') + \frac{\epsilon}{2}$, a contradiction.

References

[1] D. H. Hyers, S. M. Ulam, Approximately Convex Functions, Proceedings of the A.M.S., Vol. 3, No. 5. (1952), pp. 821-828.