Lecture 2: Brouwer and zero of inward vector fields on a convex compact set.

Philippe Bich, PSE and University Paris 1 Pantheon-Sorbonne, France.

We have seen, from Sperner:

Brouwer Theorem (particular case)

Every continuous mapping $f : \Delta_n \to \Delta_n$, where Δ_n is a n-simplex, admits a fixed point, i.e. there exists $x \in \Delta_n$ such that $f(x) = x$.

But it is true if one replaces Δ_n by any compact convex subset of $\mathbf{R}^k$ ($k > 0$ given).

Brouwer Theorem (general version)

Every continuous mapping $f : C \to C$, where C is a compact convex subset of $\mathbf{R}^k$ admits a fixed point, i.e. there exists $x \in C$ such that $f(x) = x$.

We will prove this general theorem in this section.

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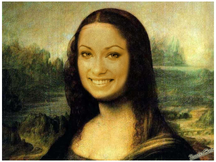

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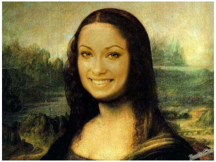

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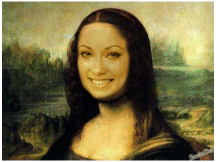

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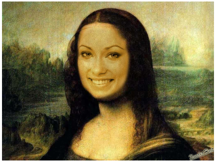

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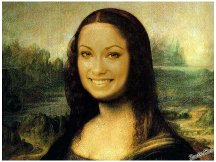

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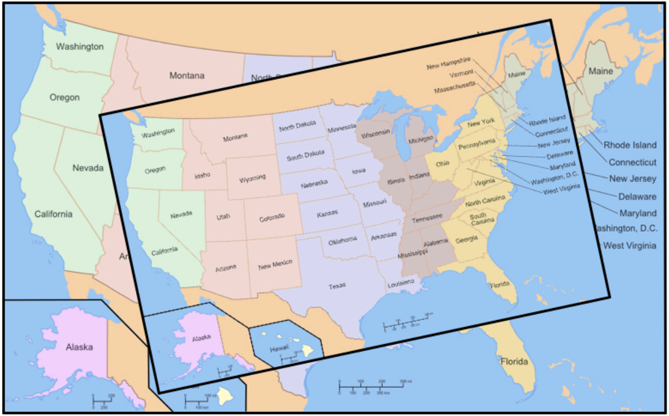
Section 2. Figure, by-products



- **Brouwer's theorem in game theory ?** could be seen as follows: $C = C_1 \times C_2 ... \times C_n$ where C_i strategy space of player i, $c = (c_1, ..., c_n)$ is a profile of strategies for each player, and $f(c) = (f_1(c), ..., f_n(c))$ where $f_i(c)$ is a strategy of player i that is the best, given the strategies $c_1, ..., c_n$ of the others.
- Then a fixed-point $f(c) = c$ is ... a Nash equilibrium.
- Problem: in general, several optimal responses to other strategies, not always possible to find one that moves continuously with respect to the others.
- Example: $C_1 = C_2 = [0, 1]$, payoff of player 1: $u_1(c_1, c_2) = c_1(\frac{1}{2} - c_2)$, payoff of player 2: $u_2(c_1, c_2) = c_2(\frac{1}{2} - c_1)$.
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Section 2. Figure, by-products: b) zero of inward vector fields.

Definition

Let B the closed unit ball of $\mathbb{R}^n$. Let $f : B \to \mathbb{R}^n$: we call it an inward vector field if for every $x \in S = \{x \in B, \|x\| = 1\}$ one has $< f(x), x > \leq 0$.

Theorem

Every continuous and inward vector field on B admits a zero, i.e., there exists $\bar{x} \in B$ such that $f(\bar{x}) = 0$.

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Every continuous and inward vector field on B admits a zero, i.e., there exists $\bar{x} \in B$ such that $f(\bar{x}) = 0$.

- The existence of a zero of an inward vector field can be easily applied to the excess demand of an economy: $c = (c_1, ..., c_n)$ is a vector of prices (where n goods), $f(c) = (f_1(c), ..., f_n(c))$ is the vector excess demand (e.g., $f_1(c) > 0$ means aggregate supply in good 1 is higher than aggregate demand in this good.)
- if one wants to incorporates markets and time, a possibility is to add some $J \times S$ matrix V in the excess demand: each column $j = 1, ..., J$ gives the payoffs of some assets in some differents states of nature $s = 1, ..., S$ tomorrow.
- In general, the returns depends on the prices of the economy, and we require some extension of the previous theorem by allowing f depends on p **and** Span $V(p)$, the vector space spanned by $V(p)$ (which appears naturally in the budget set of consumers).
- Similarly, we need some extension of Brouwer solving the equation $p = f(p, Span V(p))$. But then discontinuities: See **"An extension of Brouwer's fixed point theorem allowing discontinuities", Philippe Bich, Compte rendu à l'académie des sciences 2004]**

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In fact, Brouwer's theorem is true if one replaces Δ_n by any subset C that is homeomorphic to Δ_n:

Homeomorphic definition

A set $C \subset \mathbf{R}^k$ is homeomorphic to another set $D \subset \mathbf{R}^k$ if there exists a bijective mapping f from C to D such that f and f^{-1} are continuous.

(EXERCISE 1:) Prove that in $\mathbf{R}^1$, $]0,1[$ and $\mathbf{R}^1$ are homeomorphic. Prove that in $\mathbf{R}^k$, all the closed balls are homeomorphic.
(EXERCISE 2:) Prove that in $\mathbf{R}^2$, the unit circle and the unit ball are not homeomorphic.

Section 3. Proof of Brouwer's theorem. a) Homeorphism definition

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This is easy to prove that this theorem is true:

Brouwer Theorem

Every continuous mapping $f : C \to C$, where C is homeomorphic to Δ_n, admits a fixed point, i.e. there exists $x \in C$ such that $f(x) = x$.

(EXERCISE 3:) Prove it, using the version of Brouwer proved with Sperner's Lemma.

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Thus, to prove the version of Brouwer with C convex compact, we will prove:

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- If $C = \{x \in \mathbf{R}^n, \|x\| \leq 1\}$, it is easy to prove that $\forall x \in C, \|x\| = \inf\{\lambda > 0, \frac{x}{\lambda} \in C\}$. (EXERCISE 4:) Prove it.

Definition Let $C \subset \mathbf{R}^n$ such that $0 \in$int(C), we define p the jauge of C as follows: $\forall x \in \mathbf{R}^n, p(x) = \inf\{\lambda > 0, \frac{x}{\lambda} \in C\}$.

- **Properties: If C compact convex and $0 \in$int C, one has:**
- (1) there is $M > 0$ such that $\forall x \in E, 0 \leq p(x) \leq M\|x\|$. (EXERCISE 5:) Prove it.
- (3) $\forall t \geq 0, p(tx) = tp(x)$; (EXERCISE 6:)
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Section 3. Proof of Brouwer's theorem. d) Proof of the previous characterization.

Characterization of compact convex subsets in finite dimension

Every compact and convex subset $C \subset \mathbf{R}^n$ such that $0 \in int(C)$ is homeomorphic to $B(0,1)$, the closed unit ball of $\mathbf{R}^n$, the homeomorphism can be taken as $f : C \to B(0,1)$ defined by $f(x) = p(x).\frac{x}{\|x\|}$, where p is the jauge of C.

(EXERCISE 8: proof).

Remark that if $0 \notin int(C)$ but C has a nonempty interior one can conclude similarly.

Remark that if C has an empty interior, one can consider a smaller subspace which contains C in which C has a nonempty interior, and we can conclude similarly, replacing n by a smaller n.

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